ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{g_b g_c}{r} \le \frac{n_a w_a}{n_a - \sqrt{4r^2 + (b-c)^2}}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set $''S''$ of relations : $y^2 + z^2 = x^2(m^2 - 2n),$

$$y^3 + z^3 = x^3(m^3 - 3nn), \text{and now}, g_b g_c \overset{?}{\le} r_a w_a \overset{\substack{\text{squaring} \\ \text{and via Bogdan Fustei}}}{\Leftrightarrow}$$

$$\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\le} \frac{s(s-b)(s-c)}{s-a} \cdot \frac{4bc}{(b+c)^2} \cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\ge} (4zx + xy + yz)(4xy + zx + yz)(2x + y + z)^2$$

$$\Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4.yz - 32x^3.yz(y+z) -$$

$$5x^2.yz(y+z)^2 + 3x.yz(y^3+z^3) + 13x.y^2z^2(y+z) + 3y^2z^2(y+z)^2 \boxed{\overset{?}{\underset{(1)}{\ge}}} 0 \text{ and via}$$

set of relations $''S''$, to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2 + 4m)}^{\Omega_1} n^2 + \overbrace{(3m^3 - 5m^2 - 32m - 28)}^{\Omega_2} n + \overbrace{(8m^2 + 16m + 4)}^{\Omega_3} + \boxed{\overset{?}{\underset{(2)}{\ge}}} 0$$

and it's trivially true when : $\Omega_2 \ge 0$ and when : $\Omega_2 < 0$, then :

$$3m^3 - 5m^2 - 32m - 28 = (m-5)(3m^2 + 10m + 18) + 62 < 0$$

$$\Rightarrow (m-5)(3m^2 + 10m + 18) < -62 < 0 \Rightarrow m < 5 \rightarrow \text{(i)}$$

Now, LHS of (2) is a quadratic polynomial in $''n''$ with discriminant, $\boxed{\delta}$ = $\sigma_2^2 - 4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196)$ and when : $9m^3 - 75m^2 + 40m + 196 \le 0$, then $\delta \le 0 \Rightarrow$ (2) is trivially true and when : $9m^3 - 75m^2 + 40m + 196 > 0$, then : $9m^3 - 75m^2 + 40m + 196$

$$= \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m - 7) - \frac{14}{3} > 0$$

$$\Rightarrow \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m - 7) > \frac{14}{3} > 0 \text{ and } \because m \overset{\text{via (i)}}{\le} 5$$

$$\Rightarrow 3m(m-5) - 3m - \frac{86}{3} < 0 \therefore m < \frac{7}{3} \rightarrow \text{(ii) and so, when : } \delta > 0,$$

$$\text{then it suffices to prove : } 2\Omega_1.n \overset{?}{\le} -\Omega_2 - \sqrt{\delta} \Leftrightarrow \text{and } \because n \overset{\text{AM-GM}}{\le} \frac{m^2}{4}$$

$$\therefore \text{suffices to prove : } \Omega_1.\frac{m^2}{2} \overset{?}{\le} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2 - 2m - 14) \overset{?}{\ge} 2.\sqrt{\delta}$$

$$\boldsymbol{a}\mathbf{nd} \because 3m^2 - 2m - 14 = \left(m + \frac{5}{3}\right)(3m - 7) - \frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{it suffices to prove :}$$

$$(m+2)^4(3m^2 - 2m - 14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2 + 16m + 4) \overset{?}{\geq} 0} \therefore g_b g_c \leq r_a w_a$$

$$\Rightarrow r.\frac{n_a w_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \overset{\text{Bogdan Fustei}}{=} r.\frac{n_a w_a}{n_a - \frac{a n_a}{s}} = \frac{rs}{s-a}.w_a = r_a w_a \geq g_b g_c$$

$$\text{and so,} \frac{g_b g_c}{r} \leq \frac{n_a w_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \ \forall \Delta\, ABC,$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$