ROMANIAN MATHEMATICAL MAGAZINE

In *any* Δ ABC the fo*ll*owing re*la*tionship ho*ld*s :

$$\frac{R}{2r} \geq \frac{s\sqrt{3}.(n_a + n_b + n_c)}{(r_a + r_b + r_c)(h_a + h_b + h_c)}$$

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$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{\text{Bogdan Fustei}}{=} \sqrt{3\sum_{cyc}\left(s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \text{ and so,}$$

$$\text{it suffices to prove : } \frac{R^2}{4r^2} \overset{?}{\geq} \frac{3s^2\left(3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)\right)}{(4R+r)^2(s^2+4Rr+r^2)^2}.4R^2$$

$$\Leftrightarrow (16R^3 + 8R^2r - 431Rr^2 + 144r^3)s^4 +$$

$$r(128R^4 + 96R^3r + 2328R^2r^2 + 1154Rr^3 + 144r^4)s^2 + Rr^2(4R+r)^4 \boxed{\overset{?}{\geq}_{(1)}} 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ *and* $s^2 - (m+n) \leq 0$, where $m =$ $2R^2 + 10Rr - r^2$ & $n = 2(R-2r)\sqrt{R^2 - 2Rr}$ $\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(*)}{\leq} 0$$

(1) is trivi*all*y true if : $16R^3 + 8R^2r - 431Rr^2 + 144r^3 \geq 0$ *and* when : $16R^3 + 8R^2r - 431Rr^2 + 144r^3 < 0$, then vi*a* $(*)$, in order to prove (1),

it suffices to prove : LHS of (1) $\overset{?}{\geq}$

$$(16R^3 + 8R^2r - 431Rr^2 + 144r^3)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3)$$

$$\Leftrightarrow (16R^5 + 120R^4r - 375R^3r^2 - 1433R^2r^3 + 1224Rr^4 - 36r^5)s^2 \boxed{\overset{?}{\geq}_{(2)}}$$

$$r\left(256R^6 + 256R^5r - 6816R^4r^2 - 2864R^3r^3 + 433R^2r^4 + 324Rr^5 + 36r^6\right)$$

$\boxed{\text{Case 1}}$ $X = 16R^5 + 120R^4r - 375R^3r^2 - 1433R^2r^3 + 1224Rr^4 - 36r^5 \geq 0$

and then : LHS of (2) $\overset{\text{Gerretsen}}{\geq} X(16Rr - 5r^2) \overset{?}{\geq}$ RHS of (2)

$$\Leftrightarrow 176t^5 + 24t^4 - 2021t^3 + 2924t^2 - 780t + 16 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(176t^2(t^2-4) + 376t^2(t-2) + 187t^2 + 364t + 4(t-2)\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\boxed{\textit{Case 2}}$ $\mathbf{X < 0}$ ***and* then : LHS of ②** $\overset{\mathbf{Blundon-Gerretsen}}{\geq}$ $\mathbf{X.\frac{R(4R+r)^2}{4R-2r}} \overset{?}{\geq}$ **RHS of ②**

$$\Leftrightarrow \mathbf{256t^8 + 1024t^7 - 5536t^6 + 1968t^5 + 5569t^4 + 323t^3 + 506t^2 + 468t +}$$

$$\mathbf{72 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(4t+1)^2(16t^4 + 120t^3 + 36t^2 + 5t(t-2) + t + 18) \overset{?}{\geq} 0}$$

$\rightarrow$ **true** $\because$ $\mathbf{t \overset{Euler}{\geq} 2}$ $\Rightarrow$ ② **is true** $\therefore$ **combining both *cases*,** ② $\Rightarrow$ ① **is true** $\forall\ \Delta$ **ABC**

$$\therefore \mathbf{\frac{R}{2r} \geq \frac{s\sqrt{3}.(n_a + n_b + n_c)}{(r_a + r_b + r_c)(h_a + h_b + h_c)}}\ \forall\ \mathbf{ABC}, '' = ''\ \mathbf{iff}\ \Delta\ \mathbf{ABC\ is\ equilateral}\ \mathbf{(QED)}$$