ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+n_a+n_b+n_c}{h_a+h_b+h_c+r_a+r_b+r_c}$$

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$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{\text{Bogdan Fustei}}{=} \sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)} \therefore s\sqrt{3}+n_a+n_b+n_c \overset{CBS}{\leq} \sqrt{2}.\sqrt{3s^2+\left(\sum_{cyc} n_a\right)^2}$$

$$\leq \sqrt{2}.\sqrt{3s^2+3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)} \text{ and so, it suffices to prove :}$$

$$\frac{R}{2r} \overset{?}{\geq} \frac{6s^2+6\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}{\left(\frac{s^2+4Rr+r^2}{2R}+4R+r\right)^2} \Leftrightarrow s^4+$$

$$(16R^2-180Rr+50r^2)s^2+64R^4+96R^3r+820R^2r^2+396Rr^3+49r^4 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0$$

$$\text{and} \because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2-16Rr+5r^2)^2 \Leftrightarrow (4R^2-37Rr+10r^2)s^2+$$

$$16R^4+24R^3r+141R^2r^2+139Rr^3+6r^4 \boxed{\overset{?}{\underset{(2)}{\geq}}} 0 \text{ and it's trivially true if :}$$

$$4R^2-37Rr+10r^2 \geq 0 \text{and when : } 4R^2-37Rr+10r^2 < 0, \text{then :}$$

$$\text{LHS of (2)} \overset{\text{Blundon-Gerretsen}}{\geq}$$

$$(4R^2-37Rr+10r^2).\frac{R(4R+r)^2}{4R-2r}+16R^4+24R^3r+141R^2r^2+139Rr^3+6r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 128t^5-496t^4+384t^3+317t^2-244t-12 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2+10t(t-2)+(t^2-4)+3) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta ABC \therefore \sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+n_a+n_b+n_c}{h_a+h_b+h_c+r_a+r_b+r_c} \forall ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$