

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\sqrt{\frac{2R}{r}} \geq \frac{s\sqrt{3} + l_a + l_b + l_c + 2(n_a + n_b + n_c + r_a + r_b + r_c)}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c}$$

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$$\begin{aligned} \sum_{cyc} n_a &\stackrel{CBS}{\leq} \sqrt{3 \sum_{cyc} n_a^2} \stackrel{Bogdan Fustei}{=} \sqrt{3 \sum_{cyc} \left(s^2 - \frac{r}{R} \cdot s^2 \cdot \sec^2 \frac{A}{2} \right)} \\ &= \sqrt{3 \left(\frac{(3R-r)s^2 - r(4R+r)^2}{R} \right)} \therefore s\sqrt{3} + n_a + n_b + n_c \stackrel{CBS}{\leq} \sqrt{2} \cdot \sqrt{3s^2 + \left(\sum_{cyc} n_a \right)^2} \\ &\leq \sqrt{2} \cdot \sqrt{3s^2 + 3 \left(\frac{(3R-r)s^2 - r(4R+r)^2}{R} \right)} \text{ and so, it suffices to prove :} \end{aligned}$$

$$\frac{R}{2r} \stackrel{?}{\geq} \frac{6s^2 + 6 \left(\frac{(3R-r)s^2 - r(4R+r)^2}{R} \right)}{\left(\frac{s^2 + 4Rr + r^2}{2R} + 4R + r \right)^2} \Leftrightarrow s^4 +$$

$$(16R^2 - 180Rr + 50r^2)s^2 + 64R^4 + 96R^3r + 820R^2r^2 + 396Rr^3 + 49r^4 \stackrel{?}{\geq} 0$$

and $\therefore (s^2 - 16Rr + 5r^2)^2 \stackrel{Gerretsen}{\geq} 0 \therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (4R^2 - 37Rr + 10r^2)s^2 +$$

$$16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \stackrel{?}{\geq} 0 \text{ and it's trivially true if :}$$

$$4R^2 - 37Rr + 10r^2 \geq 0 \text{ and when : } 4R^2 - 37Rr + 10r^2 < 0, \text{ then :}$$

$$\begin{aligned} \text{LHS of ②} &\stackrel{Blundon-Gerretsen}{\geq} \\ (4R^2 - 37Rr + 10r^2) \cdot \frac{R(4R+r)^2}{4R-2r} + 16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 &\stackrel{?}{\geq} 0 \end{aligned}$$

$$\Leftrightarrow 128t^5 - 496t^4 + 384t^3 + 317t^2 - 244t - 12 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2 + 10t(t-2) + (t^2-4) + 3) \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore t \stackrel{Euler}{\geq} 2$$

$$\Rightarrow ② \Rightarrow ① \text{ is true } \forall \Delta ABC \therefore \sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3} + n_a + n_b + n_c}{h_a + h_b + h_c + r_a + r_b + r_c} \forall ABC \rightarrow (i)$$

$$\begin{aligned} \text{and now, } (b+c)^2 &\stackrel{?}{\geq} 32Rr \cdot \cos^2 \frac{A}{2} = 8r(r_b + r_c) = 8r^2s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= 8(s-a)(s-b)(s-c) \cdot \frac{a}{(s-b)(s-c)} = 4a(b+c-a) \end{aligned}$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \stackrel{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore b+c \geq \sqrt{32Rr} \cdot \cos \frac{A}{2} \Rightarrow m_a \stackrel{\text{Lascu}}{\geq} \frac{b+c}{2} \cdot \cos \frac{A}{2} \geq \sqrt{8Rr} \cdot \cos^2 \frac{A}{2}$$

$$\text{and analogs} \Rightarrow \sum_{\text{cyc}} m_a \geq \sqrt{8Rr} \cdot \frac{4R+r}{2R} = \sqrt{\frac{2r}{R}} \cdot (4R+r)$$

$$\therefore \sqrt{\frac{R}{2r}} \cdot (m_a + m_b + m_c) \geq r_a + r_b + r_c \rightarrow \text{(ii) and finally,}$$

$$l_a + l_b + l_c \stackrel{\text{AM-GM}}{\leq} \sum_{\text{cyc}} \sqrt{s(s-a)} \stackrel{\text{CBS}}{\leq} \sqrt{3s} \cdot \sqrt{\sum_{\text{cyc}} (s-a)} \Rightarrow l_a + l_b + l_c \leq s\sqrt{3} \rightarrow \text{(iii)}$$

$$\therefore \text{(iii)} \Rightarrow \frac{s\sqrt{3} + l_a + l_b + l_c + 2(n_a + n_b + n_c + r_a + r_b + r_c)}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c} \leq$$

$$\frac{2((s\sqrt{3} + n_a + n_b + n_c) + r_a + r_b + r_c)}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c} \stackrel{\text{via (i)}}{\leq}$$

$$\frac{2\left(\sqrt{\frac{R}{2r}} \cdot (h_a + h_b + h_c + r_a + r_b + r_c)\right) + 2(r_a + r_b + r_c)}{h_a + h_b + h_c + r_a + r_b + r_c + m_a + m_b + m_c} \stackrel{?}{\leq} \sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow \sqrt{\frac{2R}{r}} \cdot (m_a + m_b + m_c) \stackrel{?}{\geq} 2(r_a + r_b + r_c) \rightarrow \text{true via (ii) and so,}$$

$$\sqrt{\frac{2R}{r}} \geq \frac{s\sqrt{3} + l_a + l_b + l_c + 2(n_a + n_b + n_c + r_a + r_b + r_c)}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c} \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)