ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $fo\ell\ell owing$ $re\ell ationship$ $ho\ell ds$:

$$\sqrt{\frac{R}{2r}} \geq \frac{r_a + r_b + r_c + \ell_a + \ell_b + \ell_c + n_a + n_b + n_c}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c}$$

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$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{Bogdan\ Fustei}{=} \sqrt{3\sum_{cyc}\left(s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \therefore s\sqrt{3} + n_a + n_b + n_c \overset{CBS}{\leq} \sqrt{2}.\sqrt{3s^2 + \left(\sum_{cyc} n_a\right)^2}$$

$$\leq \sqrt{2}.\sqrt{3s^2 + 3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \text{ and so, it suffices to prove :}$$

$$\frac{R}{2r} \overset{?}{\geq} \frac{6s^2 + 6\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)}{\left(\frac{s^2+4Rr+r^2}{2R} + 4R + r\right)^2} \Leftrightarrow s^4 +$$

$$(16R^2 - 180Rr + 50r^2)s^2 + 64R^4 + 96R^3r + 820R^2r^2 + 396Rr^3 + 49r^4 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0$$

$$and \because (s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (4R^2 - 37Rr + 10r^2)s^2 +$$

$$16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \boxed{\overset{?}{\underset{(2)}{\geq}}} 0 \text{ and it's trivially true if :}$$

$$4R^2 - 37Rr + 10r^2 \geq 0 \text{ and when : } 4R^2 - 37Rr + 10r^2 < 0, \text{then :}$$

$$\text{LHS of (2)} \overset{Blundon-Gerretsen}{\geq}$$

$$(4R^2 - 37Rr + 10r^2).\frac{R(4R+r)^2}{4R-2r} + 16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 128t^5 - 496t^4 + 384t^3 + 317t^2 - 244t - 12 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2 + 10t(t-2) + (t^2-4) + 3) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta \text{ ABC} \therefore \sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3} + n_a + n_b + n_c}{h_a + h_b + h_c + r_a + r_b + r_c} \ \forall \text{ ABC} \rightarrow \text{(i)}$$

$$\mathit{and\ now},(\mathbf{b}+\mathbf{c})^2\overset{?}{\ge}32\mathrm{Rr}.\cos^2\frac{A}{2}=8\mathrm{r}(\mathrm{r_b}+\mathrm{r_c})=8\mathrm{r}^2\mathrm{s}\left(\frac{1}{\mathrm{s}-\mathrm{b}}+\frac{1}{\mathrm{s}-\mathrm{c}}\right)$$

$$=8(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c}).\frac{a}{(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}=4a(\mathrm{b}+\mathrm{c}-a)$$

$$\Leftrightarrow(\mathrm{b}+\mathrm{c})^2+4a^2-4a(\mathrm{b}+\mathrm{c})\overset{?}{\ge}0\Leftrightarrow(\mathrm{b}+\mathrm{c}-2a)^2\overset{?}{\ge}0$$

$$\rightarrow\text{true}\therefore\mathrm{b}+\mathrm{c}\ge\sqrt{32\mathrm{Rr}}.\cos\frac{A}{2}\Rightarrow\mathrm{m}_a\overset{Lascu}{\ge}\frac{\mathrm{b}+\mathrm{c}}{2}.\cos\frac{A}{2}\ge\sqrt{8\mathrm{Rr}}.\cos^2\frac{A}{2}$$

$$\mathit{and\ analogs}\Rightarrow\sum_{\mathrm{cyc}}\mathrm{m}_a\ge\sqrt{8\mathrm{Rr}}.\frac{4\mathrm{R}+\mathrm{r}}{2\mathrm{R}}=\sqrt{\frac{2\mathrm{r}}{\mathrm{R}}}.(4\mathrm{R}+\mathrm{r})$$

$$\therefore\sqrt{\frac{\mathrm{R}}{2\mathrm{r}}}.(\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c})\ge\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}\rightarrow\text{(ii)}\ \mathit{and\ finally},$$

$$l_a+l_\mathrm{b}+l_\mathrm{c}\overset{\mathrm{AM-GM}}{\le}\sum_{\mathrm{cyc}}\sqrt{\mathrm{s}(\mathrm{s}-a)}\overset{\mathrm{CBS}}{\le}\sqrt{3\mathrm{s}}.\sqrt{\sum_{\mathrm{cyc}}(\mathrm{s}-a)}\Rightarrow l_a+l_\mathrm{b}+l_\mathrm{c}\le\mathrm{s}\sqrt{3}\rightarrow\text{(iii)}$$

$$\therefore\text{(iii)}\Rightarrow\frac{\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}+l_a+l_\mathrm{b}+l_\mathrm{c}+\mathrm{n}_a+\mathrm{n_b}+\mathrm{n_c}}{\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c}+\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c}+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}}\le$$

$$\frac{\left(\mathrm{s}\sqrt{3}+\mathrm{n}_a+\mathrm{n_b}+\mathrm{n_c}\right)+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}}{\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c}+\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c}+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}}\overset{\mathit{via}\ \text{(i)}}{\le}$$

$$\frac{\left(\sqrt{\frac{\mathrm{R}}{2\mathrm{r}}}.(\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c}+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c})\right)+(\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c})}{\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c}+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}+\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c}}\overset{?}{\le}\sqrt{\frac{\mathrm{R}}{2\mathrm{r}}}$$

$$\Leftrightarrow\sqrt{\frac{\mathrm{R}}{2\mathrm{r}}}.(\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c})\overset{?}{\ge}\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}\rightarrow\text{true}\ \mathit{via}\ \text{(ii)}\ \mathit{and}\ \text{so},$$

$$\sqrt{\frac{\mathrm{R}}{2\mathrm{r}}}\ge\frac{\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}+l_a+l_\mathrm{b}+l_\mathrm{c}+\mathrm{n}_a+\mathrm{n_b}+\mathrm{n_c}}{\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c}+\mathrm{m}_a+\mathrm{m_b}+\mathrm{m_c}+\mathrm{r}_a+\mathrm{r_b}+\mathrm{r_c}}\ \forall\,\Delta\,\mathrm{ABC},$$

$$''=''\ \text{iff}\ \Delta\,\mathrm{ABC}\ \text{is equilateral (QED)}$$