ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)\geq\frac{\sum_{cyc}n_a+\sum_{cyc}\left(p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)=\sum_{cyc}\left(\frac{rs.a(b+c)}{s(s-a).2bc\cos\frac{A}{2}}\cdot\sqrt{\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}.s\tan\frac{A}{2}}}\right)$$

$$=\sqrt{\frac{r}{2R}}\cdot\sum_{cyc}\frac{ra(b+c)}{(s-a)bc.2\cos\frac{A}{2}\sin\frac{A}{2}}=\sqrt{\frac{r}{2R}}\cdot\sum_{cyc}\frac{2Rr.a(b+c)}{(s-a)abc}$$

$$=\sqrt{\frac{r}{2R}}\cdot\frac{2Rr}{4Rrs}\cdot\sum_{cyc}\frac{a(s+s-a)}{s-a}=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}\cdot\left(s\sum_{cyc}\frac{a-s+s}{s-a}+2s\right)$$

$$=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}\cdot\left(s\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+2s\right)=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}.s.\frac{4R}{r}$$

$$\Rightarrow\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)=\sqrt{\frac{2R}{r}}\rightarrow (1)\text{ and again, }p_a\cdot\sqrt{\frac{w_a}{g_a}}\leq n_a\text{ and analogs}$$

$$\left(\begin{array}{c}\text{Reference : Solution to Inequality in Triangle}-118\text{ by Bogdan Fustei;}\\ \text{publshed at www.ssmrmh.ro}\end{array}\right)$$

$$\Rightarrow\frac{\sum_{cyc}n_a+\sum_{cyc}\left(p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}\leq\frac{2\sum_{cyc}n_a}{4R+r}\overset{CBS}{\leq}\frac{2.\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{2.\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}=\frac{2.\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)$$

$$\overset{\text{via }(1)}{=}\sqrt{\frac{2R}{r}}\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2+6r^2(4R+r)^2\boxed{\overset{?}{\underset{(*)}{\geq}}}6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{\text{Rouche}}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\text{and}\because R-2r\overset{\text{Euler}}{\geq}0\therefore\text{ it suffices to prove following squaring :}$$

$$\mathbf{R^2(16R^2+4Rr-63r^2)^2 \overset{?}{\geq} 144r^2(3R-r)^2(R^2-2Rr)}$$

$$\Leftrightarrow \mathbf{256t^5+128t^4-3296t^3+2952t^2+2097t+288 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)}$$

$$\Leftrightarrow \mathbf{\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right) \overset{?}{\geq} 0 \to true}$$

$$\mathbf{(strict\ inequality) \because t \overset{Euler}{\geq} 2 \Rightarrow (*)\ is\ true}$$

$$\therefore \sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right) \geq \frac{\sum_{cyc} n_a + \sum_{cyc}\left(p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}\ \forall\ ABC,$$

$$'' = ''\ \mathbf{iff}\ \Delta\ \mathbf{ABC\ is\ equilateral\ (QED)}$$