ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{p_a\cdot\sqrt{\frac{w_a}{g_a}}+p_b\cdot\sqrt{\frac{w_b}{g_b}}+p_c\cdot\sqrt{\frac{w_c}{g_c}}}{r_a+r_b+r_c}\le\left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_aw_bw_c}{h_ah_bh_c}\right)^{\frac{1}{3}}\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^{\frac{1}{3}}}{2r}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{R}{2r}\right)\cdot\left(\frac{w_aw_bw_c}{h_ah_bh_c}\right)^2\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^2}{64r^6}=$$

$$=\frac{\left(\frac{R}{2r}\right)}{64r^6}\cdot\left(\frac{32Rr^2s^3}{(a+b)(b+c)(c+a)}\cdot\frac{R}{2r^2s^2}\cdot\frac{r^3(a+b)(b+c)(c+a)}{4Rrs}\right)^2=\frac{R^3}{8r^3}$$

$$\therefore \text{RHS}=\sqrt{\frac{R}{2r}}\to ① \textit{ and } \because p_a\cdot\sqrt{\frac{w_a}{g_a}}\le n_a \textit{ and analogs}$$

(Reference : Solution to Inequality in Triangle – 118 by Bogdan Fustei; published at www.ssmrmh.ro)

$$\therefore\frac{\sum_{cyc}\left(p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}\le\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\le}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\le}\text{RHS}\overset{\text{via }①}{=}\sqrt{\frac{R}{2r}}$$

$$\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2+6r^2(4R+r)^2\boxed{\overset{?}{\underset{(*)}{\ge}}}6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{\text{Rouche}}{\le}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\ge}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\textit{and}\because R-2r\overset{\text{Euler}}{\ge}0\ \therefore\text{ it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\ge}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\ge}0\to\text{true}$$

$$(\text{strict inequality})\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{p_a\cdot\sqrt{\frac{w_a}{g_a}}+p_b\cdot\sqrt{\frac{w_b}{g_b}}+p_c\cdot\sqrt{\frac{w_c}{g_c}}}{r_a+r_b+r_c}\le\left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_aw_bw_c}{h_ah_bh_c}\right)^{\frac{1}{3}}\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^{\frac{1}{3}}}{2r}$$

$\forall$ ABC, $''=''$ iff Δ ABC is equilateral (QED)