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In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r}\geq\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}$$

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$$\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\leq}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{Bogdan\ Fustei}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\sqrt{\frac{R}{2r}}$$

$$\overset{squaring}{\Leftrightarrow}\ R^2(4R+r)^2+6r^2(4R+r)^2\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{Rouche}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$and\ \because R-2r\overset{Euler}{\geq}0\ \therefore\text{ it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\geq}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\geq}0\rightarrow\text{true}$$

$$(\text{strict inequality})\because t\overset{Euler}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore\sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r}-\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\geq$$

$$\frac{\sum_{cyc}n_a}{4r}+\frac{3s}{4r}-\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\overset{Bogdan\ Fustei}{=}\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a}{a-\frac{an_a}{s}}$$

$$=\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a(s+n_a)}{a(s-n_a)(s+n_a)}\overset{Bogdan\ Fustei}{=}\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a(s+n_a)}{a.2h_ar_a}$$

$$=\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{s+n_a}{4r}=0\ and\ so,$$

$$\sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r}\geq\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\ \forall\ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)