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In any Δ ABC the following relationship holds :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)} + \sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)}$$

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$$\text{Triangle inequality} \Rightarrow g_a \leq AI + r \overset{?}{\leq} l_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\leq} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\leq} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\leq} 1 \rightarrow \text{true} \therefore l_a \geq g_a \rightarrow \text{(i)}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\\text{and}\\\text{Ajiba}}}{\Leftrightarrow} \left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq}$$

$$\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2 + \frac{(b-c)^4}{324} +$$

$$\frac{(b-c)^2}{9}.\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \; (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$

$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \to \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2+\frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \to (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a l_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2.m_a l_a$$

$$\Rightarrow n_a \geq p_a.\sqrt{\frac{l_a}{g_a}} \Rightarrow \boxed{t \leq n_a} \left(t = p_a.\sqrt{\frac{l_a}{g_a}}\right)$$

$$\Rightarrow \left(\sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)}+\sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)}\right)^2 = \frac{r_a}{t}+\frac{t}{r_a}+2 = \frac{r_a^2+t^2}{r_a.p_a.\sqrt{\frac{l_a}{g_a}}}+2 \leq$$

$$\frac{r_a^2+n_a^2}{r_a.h_a.\sqrt{\frac{l_a}{g_a}}}+2 \overset{\substack{\text{via (i)}\\ \text{and}\\ \text{via Bogdan Fustei}}}{\leq} \frac{s^2\tan^2\frac{A}{2}+s^2-2r_a h_a}{r_a h_a}+2 = \frac{2s^2\sec^2\frac{A}{2}}{\frac{r}{R}.s^2.\sec^2\frac{A}{2}} = \frac{2R}{r}$$

$$\Rightarrow \sqrt{\frac{2R}{r}} \geq \sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)}+\sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)} \ \forall \Delta ABC, ''='' \text{ iff } b=c \text{ (QED)}$$