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In any Δ ABC the following relationship holds :

$$7 \geq \sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)+\sum_{cyc}\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

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$$\text{Triangle inequality} \Rightarrow g_a \leq AI + r \overset{?}{\leq} l_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\leq} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\leq} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\leq} 1 \rightarrow \text{true} \therefore l_a \geq g_a \rightarrow (1)$$

$$\text{Again}, \sum_{cyc}\frac{AI}{l_a} = \sum_{cyc}\frac{r(b+c)}{\sin\frac{A}{2}.2bc\cos\frac{A}{2}} = \sum_{cyc}\frac{2R.r(b+c)}{abc}$$

$$= \frac{2Rr}{4Rrs}.\sum_{cyc}(b+c) = \frac{8Rrs}{4Rrs} = 2 \therefore \sum_{cyc}\frac{AI}{l_a} = 2 \rightarrow (2)$$

$$\text{Now}, \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \overset{\text{Bogdan Fustei}}{\geq} \sum_{cyc}\frac{n_a-AI+r-g_a}{l_a}$$

(Reference : Inequality in Triangle by Bogdan Fustei − 103; published at www.ssmrmh.ro)

$$\overset{\text{Triangle inequality}}{\geq} \sum_{cyc}\frac{n_a-AI+r-AI-r}{l_a} = \sum_{cyc}\frac{n_a}{l_a} - \sum_{cyc}\frac{2AI}{l_a} \overset{\text{via }(2)}{=} \sum_{cyc}\frac{n_a}{l_a} - 4$$

$$\Rightarrow 7 + \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \geq 3 + \sum_{cyc}\frac{n_a}{l_a} = \sum_{cyc}\frac{n_a+l_a}{l_a}$$

$$\overset{\text{via }(1)\text{ and analogs}}{\geq} \sum_{cyc}\frac{n_a+g_a}{l_a} \overset{\text{Bogdan Fustei (2019)}}{\geq} \sum_{cyc}\frac{2m_a}{l_a} \overset{\text{Lascu}}{\geq} \sum_{cyc}\frac{(b+c)\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}$$

$$= \sum_{cyc}\frac{(b+c)^2}{2bc} \therefore 7 + \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \boxed{\overset{(*)}{\geq}} \sum_{cyc}\frac{(b+c)^2}{2bc}$$

We shall now prove that : $\frac{(b+c)^2}{2bc} \overset{?}{\geq} \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \cdot \sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}$

$$\Leftrightarrow \frac{\left(x^2+\frac{1}{x^2}+2\right)}{2} \overset{?}{\geq} \left(x+\frac{1}{x}\right) \cdot \sqrt[4]{\frac{1}{2}\left(x+\frac{1}{x}\right)} \left(x=\sqrt{\frac{b}{c}}\right)$$

$$\Leftrightarrow \frac{(x^2+1)^8}{16x^8} \overset{?}{\geq} \frac{(x^2+1)^4}{x^4} \cdot \frac{x^2+1}{2x} \Leftrightarrow (x^2+1)^3 \overset{?}{\geq} 8x^3 \Leftrightarrow x^2+1 \overset{?}{\geq} 2x$$

$\Leftrightarrow (x-1)^2 \overset{?}{\geq} 0 \to$ true $\therefore \frac{(b+c)^2}{2bc} \overset{?}{\geq} \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \cdot \sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}$ *and analogs*

$\Rightarrow \sum_{cyc} \frac{(b+c)^2}{2bc} \boxed{\overset{(**)}{\geq}} \sum_{cyc} \left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \cdot \sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)$ *and* so, $(*)$ *and* $(**) \Rightarrow$

$$7+\sum_{cyc} \frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \geq \sum_{cyc} \left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \cdot \sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)$$

$$\Rightarrow 7 \geq \sum_{cyc} \left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \cdot \sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)+\sum_{cyc} \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

$\forall \Delta$ ABC, $'' ='' $ iff Δ ABC is equi*lateral* (QED)