ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC prove that the following relationship holds:

$$\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}\ge\frac{n_a+n_b+n_c-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}$$

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Solution by Jenish Rijal-Nepal

Let I be the incenter, D be the projection of I on BC, and T_a be the A-excircle tangency point on BC. Then: $ID=r$, $BD=s-b$, $BT_a=s-c$, and $AT_a=n_a$.

Since $DT_a=|(s-c)-(s-b)|=|b-c|$, Pythagoras on $\triangle IDT_a$ yields:

$$IT_a=\sqrt{r^2+(b-c)^2}$$

By the triangle inequality on $\triangle AIT_a$, we have:

$$n_a\le AI+IT_a\Rightarrow n_a-\sqrt{(b-c)^2+r^2}\le AI$$

Summing cyclically for all three vertices, we have:

$$\sum_{cyc}n_a-\sum_{cyc}\sqrt{(b-c)^2+r^2}\overbrace{\le}^{①}\sum_{cyc}AI$$

Via $AI=\frac{r}{\sin\frac{A}{2}}$ & $s=r\sum_{cyc}\cot\frac{A}{2}$ and we also

subtract s from both sides of ① & divide by r:

$$\frac{\sum_{cyc}n_a-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}\le\sum_{cyc}\left(\frac{1}{\sin\frac{A}{2}}-\cot\frac{A}{2}\right)$$

Via half $-$ angle identity $\frac{1}{\sin\frac{A}{2}}-\cot\frac{A}{2}=\tan\frac{A}{4}$, we have:

$$\frac{n_a+n_b+n_c-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}\le\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}$$

Equality holds if and only if the triangle is equilateral.