ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$7 \geq \sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{p_a}{l_a}}}}}\right)$$

$$+\sum_{cyc}\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \leq AI + r \overset{?}{\leq} l_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\leq} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\leq} \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\leq} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\leq} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\leq} 1 \rightarrow \text{true} \therefore l_a \geq g_a \rightarrow (1)$$

$$\text{Again, } \sum_{cyc}\frac{AI}{l_a} = \sum_{cyc}\frac{r(b+c)}{\sin\frac{A}{2}.2bc\cos\frac{A}{2}} = \sum_{cyc}\frac{2R.r(b+c)}{abc}$$

$$= \frac{2Rr}{4Rrs}.\sum_{cyc}(b+c) = \frac{8Rrs}{4Rrs} = 2 \therefore \sum_{cyc}\frac{AI}{l_a} = 2 \rightarrow (2)$$

$$\text{Now, } \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \overset{\text{Bogdan Fustei}}{\geq} \sum_{cyc}\frac{n_a-AI+r-g_a}{l_a}$$

(Reference : Inequality in Triangle by Bogdan Fustei – 103; published at www.ssmrmh.ro)

$$\overset{\text{Triangle inequality}}{\geq} \sum_{cyc}\frac{n_a-AI+r-AI-r}{l_a} = \sum_{cyc}\frac{n_a}{l_a} - \sum_{cyc}\frac{2AI}{l_a} \overset{\text{via } (2)}{=} \sum_{cyc}\frac{n_a}{l_a} - 4$$

$$\Rightarrow 7 + \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \geq 3 + \sum_{cyc}\frac{n_a}{l_a} = \sum_{cyc}\frac{n_a+l_a}{l_a}$$

$$\overset{\text{via (1) and analogs}}{\geq} \sum_{cyc}\frac{n_a+g_a}{l_a} \overset{\text{Bogdan Fustei (2019)}}{\geq} \sum_{cyc}\frac{2m_a}{l_a} \rightarrow (*)$$

$$\text{Also}, \frac{2m_a}{l_a} \overset{\text{Lascu}}{\geq} \frac{(b+c)\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}} = \frac{(b+c)^2}{2bc} \Rightarrow \frac{m_a}{l_a} \boxed{\overset{(**)}{\geq}} \left(\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\right)^2 = \sigma^2\ (say)$$

$$p_a^2 l_a^2 \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ben Ajiba}}}{=} \left(s(s-a)+\frac{s(3s+a)}{(2s+a)^2}(b-c)^2\right)\left(s(s-a)-\frac{s(s-a)}{(2s-a)^2}.(b-c)^2\right)$$

$$\overset{?}{\leq} m_a^4 = s^2(s-a)^2 + \frac{(b-c)^4}{16} + \frac{s(s-a)(b-c)^2}{2} \Leftrightarrow$$

$$\left(\frac{1}{16}+\frac{s^2(3s+a)(s-a)}{(4s^2-a^2)^2}\right).(b-c)^2 + \frac{s(s-a)a\left((s-a)(16s^2+4sa)+a^3\right)}{2(4s^2-a^2)^2}$$

$$\overset{?}{\geq} 0 \to \text{true} \because s > a \Rightarrow p_a l_a \leq m_a^2 \Rightarrow \sqrt[4]{\frac{p_a}{l_a}} \leq \sqrt{\frac{m_a}{l_a}} \therefore \text{in order to prove}: \frac{2m_a}{l_a} \underbrace{\overset{?}{\geq}}_{(***)}$$

$$\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{p_a}{l_a}}}}}, \text{suffices}$$

$$\frac{2m_a}{l_a} \overset{?}{\geq} \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt{\frac{m_a}{l_a}}}}}$$

$$\Leftrightarrow \left(\frac{m_a}{l_a}\right)^{\frac{127}{128}} \overset{?}{\geq} \sigma^{\frac{85}{64}} \text{ and via } (**), \text{it suffices to prove}: \sigma^{\frac{127}{64}} \overset{?}{\geq} \sigma^{\frac{85}{64}} \Leftrightarrow \sigma^{\frac{42}{64}} \overset{?}{\geq} 1 \to \text{true}$$

$$\because \sigma = \frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) \overset{\text{AM-GM}}{\geq} 1 \overset{\text{summation }\&\ (*),(***)}{\Rightarrow} 7 + \sum_{\text{cyc}} \frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \geq$$

$$\sum_{\text{cyc}}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{p_a}{l_a}}}}}\right)$$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)