# ROMANIAN MATHEMATICAL MAGAZINE


In any $\Delta$ ABC the following relationship holds

$$\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\geq$$

$$\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)}$$


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$$s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)$$

$$=\sum_{cyc}\frac{h_a(s^2-2r_ah_a)}{s+\sqrt{2r_ah_a}}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{n_a^2}{\frac{s}{h_a}+\sqrt{\frac{2r_a}{h_a}}}\overset{\text{Bergstrom}}{\geq}\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sum_{cyc}\sqrt{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}}$$

$$=\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sum_{cyc}\sqrt{\frac{\sin^2\frac{A}{2}}{\frac{r}{4R}}}}\Rightarrow s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)$$

$$\overset{(1)}{\geq}\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sqrt{\frac{4R}{r}}.\left(\sum_{cyc}\sin\frac{A}{2}\right)}\ \textit{and again},\ \sum_{cyc}\frac{l_b+lc}{h_a}=\sum_{cyc}\left(l_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)$$

$$=\sum_{cyc}\left(\frac{2bc}{b+c}.\cos\frac{A}{2}.\left(\frac{b+c}{2rs}\right)\right)=\frac{4Rrs}{rs}.\sum_{cyc}\frac{\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}=\sum_{cyc}\operatorname{cosec}\frac{A}{2}$$

$$\Rightarrow\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\overset{(2)}{=}\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}$$

Now, (1) *and* (2)

$$\Rightarrow\left(\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\right)\left(\begin{array}{c}s(h_a+h_b+h_c)-\\\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)\end{array}\right)\geq$$

$$\left(\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sqrt{\frac{4R}{r}}.\left(\sum_{cyc}\sin\frac{A}{2}\right)}\right).\left(\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}\right)\overset{?}{\geq}(n_a+n_b+n_c)^2$$

$$\Leftrightarrow 2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)\overset{?}{\geq}\frac{4R}{r}\left(\frac{2R-r}{2R}+2.\frac{r}{4R}.\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)\to\text{true}$$

$$\therefore \frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\geq$$

$$\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)}$$

$\forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$