ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3R}{r}-2\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3\sqrt[3]{abc}}{a+b+c}\ge 4$$

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$$Lemma: \forall\, x,y,z>0\,,\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{(x+y+z)}{\sqrt[3]{xyz}}$$

$$Proof: 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)=\left(\frac{x}{y}+\frac{x}{y}+\frac{y}{z}\right)+\left(\frac{y}{z}+\frac{y}{z}+\frac{z}{x}\right)+\left(\frac{z}{x}+\frac{z}{x}+\frac{x}{y}\right)\overset{AM-GM}{\ge}$$

$$\ge\sum\frac{3x}{\sqrt[3]{xyz}}=\frac{3(x+y+z)}{\sqrt[3]{xyz}}\;\; or\;\; \frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{(x+y+z)}{\sqrt[3]{xyz}}\;\;(proof\ complete)$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3\sqrt[3]{abc}}{a+b+c}=\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{3\sqrt[3]{abc}}{a+b+c}\ge$$

$$\overset{\substack{AM-GM\\Lemma}}{\ge}\frac{2}{3}\times 3+\frac{1}{3}\cdot\frac{a+b+c}{\sqrt[3]{abc}}+\frac{3\sqrt[3]{abc}}{a+b+c}\overset{AM-GM}{\ge}2+2\sqrt{\frac{1}{3}\cdot\frac{a+b+c}{\sqrt[3]{abc}}\cdot\frac{3\sqrt[3]{abc}}{a+b+c}}=4$$

$$\frac{3\sqrt[3]{abc}}{a+b+c}\overset{AM-GM}{\le}\frac{a+b+c}{a+b+c}=1\ \&\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{CBS}{\le}$$

$$\le\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}\overset{Leibniz\,,Steining}{\le}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$We\ need\ to\ show: 1+\frac{3R}{2r}\le\frac{3R}{r}-2\ or,\frac{3R}{2r}\ge 3\ or, R\ge 2r\ true\ by\ Euler.$$

Equality holds for an equilateral triangle.