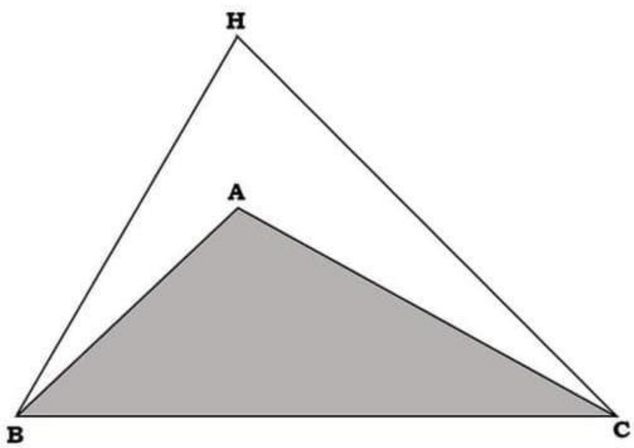
ROMANIAN MATHEMATICAL MAGAZINE

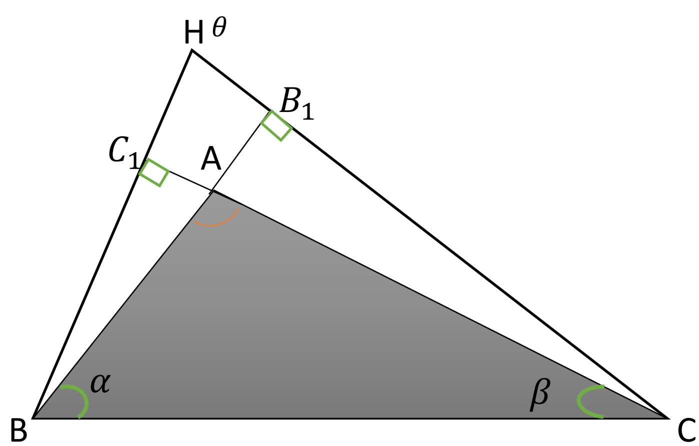
**In the figure below ,the point H is the orthocenter of the triangle ABC ,whose sides fulfill the condition :**

$\frac{AC}{AB}=\frac{BC}{AC}=\sqrt{2}$ **.That said , show that** $\frac{[HBAC]}{[ABC]}=\frac{8}{7}$




***Proposed by Rachid Iksi-Morocco***

***Solution by Mirsadix Muzefferov-Azerbaijan***



$$In\ \Delta ABC\ law\ cosine:\ \cos\theta=\frac{AB^2+AC^2-BC^2}{2AB\cdot AC}=\frac{\frac{BC^2}{4}+\frac{BC^2}{2}-BC^2}{2\cdot\frac{BC}{\sqrt{2}}\cdot\frac{BC}{2}}$$

$$AB=\frac{AC}{\sqrt{2}},AC=\frac{BC}{\sqrt{2}},AB=\frac{BC}{2}$$

$$\cos\theta=\frac{(1+2-4)\sqrt{2}}{4}=-\frac{\sqrt{2}}{4}$$

$$S(ABC)=\frac{1}{2}BC^2\cdot\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}=\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}$$

$$S(HBC)=\frac{1}{2}BC^2\frac{\sin(\frac{\pi}{2}-\beta)\sin(\frac{\pi}{2}-\alpha)}{\sin(\pi-(\alpha+\beta))}=\frac{1}{2}BC^2\frac{\cos\alpha\cdot\cos\beta}{\sin\theta}$$

$$\frac{S(HBAC)}{S(ABC)}=\frac{\frac{1}{2}BC^2\frac{\cos\alpha\cdot\cos\beta}{\sin\theta}-\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}}{\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}}=\frac{\cos(\alpha+\beta)}{\sin\alpha\sin\beta}=-\frac{\cos\theta}{\sin\alpha\sin\beta}$$

$$In\ \Delta ABC\ law\ sine:\ \frac{BC}{\sin\theta}=\frac{AC}{\sin\alpha}\Rightarrow\frac{BC}{AC}=\frac{\sin\theta}{\sin\alpha}=\sqrt{2}\Rightarrow\sin\alpha=\frac{\sin\theta}{\sqrt{2}}$$

$$\frac{AC}{\sin\alpha}=\frac{AB}{\sin\beta}\Rightarrow\frac{AC}{AB}=\frac{\sin\alpha}{\sin\beta}=\sqrt{2}\Rightarrow\sin\beta=\frac{\sin\alpha}{\sqrt{2}}=\frac{\sin\theta}{2}$$

$$\frac{S(HBAC)}{S(ABC)}=-\frac{\cos\theta}{\frac{\sin\theta}{\sqrt{2}}\cdot\frac{\sin\theta}{2}}=-2\sqrt{2}\cdot\frac{\cos\theta}{sin^2\theta}=-2\sqrt{2}\cdot\frac{-\frac{\sqrt{2}}{4}}{\frac{7}{8}}=\frac{8}{7}$$

$$sin^2\theta=1-cos^2\theta=1-\frac{2}{16}=\frac{7}{8}$$