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If in $\triangle ABC$ with $D \in (AC), E \in (BC)$ such that $AB = CD, DB = DE$

and $\angle ABD = \angle EDC = 2 \cdot \angle ACB$, then find the angle measures of triangle ABC

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$$\begin{aligned}\angle C &= \alpha \Rightarrow \angle ABD = \angle EDC = 2\alpha \\ \angle DEB &= \angle C + \angle EDC = \alpha + 2\alpha = 3\alpha \\ DB &= DE \Rightarrow \angle DBE = \angle DEB = 3\alpha \\ \angle B &= \angle ABD + \angle DBE = 2\alpha + 3\alpha = 5\alpha \\ \angle BDA &= \angle C + \angle DBC = \alpha + 3\alpha = 4\alpha \\ \angle A &= 180^\circ - (\angle B + \angle C) = 180^\circ - 6\alpha\end{aligned}$$

$$\text{Law of Sines in } \triangle ABD: \frac{AB}{\sin(4\alpha)} = \frac{BD}{\sin(180^\circ - 6\alpha)} \Rightarrow AB = BD \frac{\sin(4\alpha)}{\sin(6\alpha)}$$

$$\text{Law of Sines in } \triangle BCD: \frac{CD}{\sin(3\alpha)} = \frac{BD}{\sin(\alpha)} \Rightarrow CD = BD \frac{\sin(3\alpha)}{\sin(\alpha)}$$

$$\begin{aligned}AB &= CD \\ BD \frac{\sin(4\alpha)}{\sin(6\alpha)} &= BD \frac{\sin(3\alpha)}{\sin(\alpha)} \\ \sin(4\alpha)\sin(\alpha) &= \sin(6\alpha)\sin(3\alpha) \\ \frac{1}{2}[\cos(3\alpha) - \cos(5\alpha)] &= \frac{1}{2}[\cos(3\alpha) - \cos(9\alpha)] \\ \cos(5\alpha) &= \cos(9\alpha)\end{aligned}$$

$$\begin{aligned}\angle A > 0 &\Rightarrow 180^\circ - 6\alpha > 0 \Rightarrow \alpha < 30^\circ \\ 9\alpha + 5\alpha &= 360^\circ \Rightarrow 14\alpha = 360^\circ \Rightarrow \alpha = \frac{180^\circ}{7}\end{aligned}$$

$$\begin{aligned}\angle A &= 180^\circ - 6\left(\frac{180^\circ}{7}\right) = \frac{180^\circ}{7} = \frac{\pi}{7} \\ \angle B &= 5\left(\frac{180^\circ}{7}\right) = \frac{900^\circ}{7} = \frac{5\pi}{7} \\ \angle C &= \frac{180^\circ}{7} = \frac{\pi}{7}\end{aligned}$$