

ROMANIAN MATHEMATICAL MAGAZINE

In any $\triangle ABC$ the following relationship holds:

$$g_a^2 = 3r^2 + \frac{r^2(R+r)}{R\sin^2 \frac{A}{2}} = 2(R+r)h_a - r(4R+r) = \frac{(h_a - 2r)(4r_a + r_b + r_c)}{2}$$

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Here, by Stewart's theorem:

$$g_a^2 = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a} = (s-a)^2 + \frac{4\Delta^2}{as}.$$

Here, $\frac{\Delta}{s} = r$ and $\frac{2\Delta}{a} = h_a$, so we have:

$$\boxed{g_a^2 = (s-a)^2 + 2rh_a.}$$

Via $2Rh_a = bc$, $4Rr = \frac{abc}{s}$, and $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$, we have:

$$\begin{aligned} 2(R+r)h_a - r(4R+r) &= 2Rh_a - 4Rr - r^2 + 2rh_a = bc - \frac{abc}{s} - r^2 + 2rh_a = \\ &= \frac{s-a}{s} [bc - (s-b)(s-c)] + 2rh_a = \frac{s-a}{s} [s(s-a)] + 2rh_a = (s-a)^2 + 2rh_a = g_a^2 \end{aligned}$$

$$\therefore \boxed{2(R+r)h_a - r(4R+r) = g_a^2.}$$

Via $\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}$ and $bc = 2Rh_a$:

$$R\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{2h_a} \Rightarrow \frac{r^2(R+r)}{R\sin^2 \frac{A}{2}} = 2(R+r)h_a \left(\frac{r^2}{(s-b)(s-c)} \right)$$

As $\frac{r^2}{(s-b)(s-c)} = \frac{s-a}{s} = 1 - \frac{a}{s}$ and $ah_a = 2rs$, we get:

$$3r^2 + 2(R+r)h_a \left(1 - \frac{a}{s} \right) = 3r^2 + 2(R+r)h_a - \frac{2(R+r)(2rs)}{s} =$$

ROMANIAN MATHEMATICAL MAGAZINE

$$= 3r^2 + 2(R + r)h_a - 4r(R + r) = 2(R + r)h_a - r(4R + r).$$

$$\therefore \boxed{3r^2 + \frac{r^2(R + r)}{R \sin^2 \frac{A}{2}} = 2(R + r)h_a - r(4R + r).}$$

At last, via $r_a + r_b + r_c = 4R + r$ and $r_a(h_a - 2r) = rh_a$, we have

$$\begin{aligned} \frac{(h_a - 2r)(4r_a + r_b + r_c)}{2} &= \frac{(h_a - 2r)(3r_a + 4R + r)}{2} = \\ &= \frac{3}{2}r_a(h_a - 2r) + \frac{1}{2}(h_a - 2r)(4R + r) \\ &= \frac{3}{2}rh_a + \frac{1}{2}h_a(4R + r) - r(4R + r) \\ &= 2(R + r)h_a - r(4R + r) \end{aligned}$$

$$\therefore \boxed{\frac{(h_a - 2r)(4r_a + r_b + r_c)}{2} = 2(R + r)h_a - r(4R + r).}$$