ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{In a non}-\textit{equilateral } \Delta ABC \textit{ with } \frac{2a}{R}=\frac{3\sqrt{3}b}{s}=\frac{s}{r} \textit{ prove that:}$$

$$R=\frac{14}{3}r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$Let\ \frac{2a}{R}=\frac{3\sqrt{3}b}{s}=\frac{s}{r}=k\ then\ a=\frac{kR}{2},b=\frac{ks}{3\sqrt{3}},c=kr$$

$$abc=4Rrs\ or,\frac{kR}{2}.\frac{ks}{3\sqrt{3}}.kr=4Rrs\Rightarrow k=2\sqrt{3}$$

$$a+b+c=2s\ or,\frac{kR}{2}+\frac{ks}{3\sqrt{3}}+kr=2s\ or\ \sqrt{3}R+\frac{2s}{3}+2\sqrt{3}r\overset{k=2\sqrt{3}}{=}2s$$

$$\frac{4s}{3}=\sqrt{3}(R+2r)\ or,\frac{s}{r}=\frac{3\sqrt{3}}{4}\left(\frac{R}{r}+2\right)\ (1)$$

$$ab=\frac{2\sqrt{3}}{3}Rs,bc=\frac{4\sqrt{3}}{3}rs,ca=6Rr$$

$$ab+bc+ca=s^2+r^2+4Rr\ or,\frac{2\sqrt{3}}{3}Rs+\frac{4\sqrt{3}}{3}rs+6Rr=s^2+r^2+4Rr$$

$$or,\frac{2\sqrt{3}}{3}xy+\frac{4\sqrt{3}}{3}y+6x=y^2+1+4x,\left(x=\frac{R}{r},y=\frac{s}{r}\overset{(1)}{=}\frac{3\sqrt{3}}{4}\left(\frac{R}{r}+2\right)=\frac{3\sqrt{3}}{4}(x+2)\right)$$

$$\frac{2\sqrt{3}}{3}x.\frac{3\sqrt{3}}{4}(x+2)+\frac{4\sqrt{3}}{3}.\frac{3\sqrt{3}}{4}(x+2)+6x=\left(\frac{3\sqrt{3}}{4}(x+2)\right)^2+1+4x$$

$$\frac{3}{2}x(x+2)+3(x+2)+6x=\frac{27}{16}(x+2)^2+1+4x$$

$$\frac{3}{2}x^2+12x+6=\frac{27}{16}x^2+\frac{43x}{4}+\frac{31}{4}\quad or\ \ 3x^2-20x+28=0$$

$$(3x-14)(x-2)=0$$

$$x=\frac{14}{3}\left(as\ x=\frac{R}{r}\neq 2, since\ \ triangle\ is\ \ non-equilateral\right)\ \ or\ R=\frac{14}{3}r\,.$$