

ROMANIAN MATHEMATICAL MAGAZINE

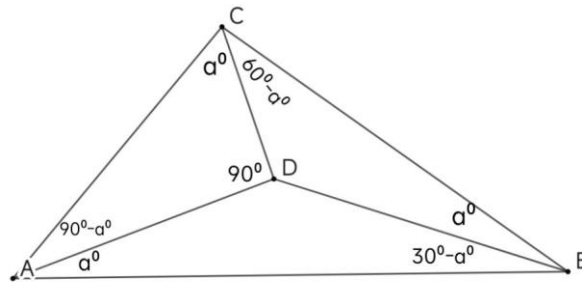
In $\triangle ABC$, let D be a point in its interior such that:

$$\angle BAD = \angle DBC = \angle DCA = \alpha, \quad \angle ADC = 90^\circ \quad \text{and} \quad AB \cdot AC = \frac{\sqrt{3}}{4} BC^2.$$

Find $\cos(\angle BAD)$.

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In $\triangle ABC$ ($\angle A = 90^\circ$), by Pythagorean theorem: $AC^2 + AB^2 = BC^2$

$$AC^2 + AB^2 = \frac{4 \cdot AB \cdot AC}{\sqrt{3}} \Rightarrow \sqrt{3} \cdot \left(\frac{AB}{AC}\right)^2 - 4 \left(\frac{AB}{AC}\right) + \sqrt{3} = 0, \quad \frac{AB}{AC} = \frac{2 \pm 1}{\sqrt{3}}$$

$$1) \frac{AB}{AC} = \sqrt{3} \quad 2) \frac{AB}{AC} = \frac{1}{\sqrt{3}}$$

$$1) \text{ Case: } \frac{AB}{AC} = \sqrt{3} \Rightarrow \angle C = 60^\circ, \angle B = 30^\circ$$

Let's use Ceva's theorem in $\triangle ABC$:

$$\sin \alpha \cdot \sin \alpha \cdot \sin \alpha = \sin(30^\circ - \alpha) \cdot \sin(60^\circ - \alpha) \cdot \sin(90^\circ - \alpha)$$

$$\sin^3 \alpha = \cos \alpha \cdot \sin(30^\circ - \alpha) \cdot \sin(60^\circ - \alpha)$$

$$\frac{\sin^3 \alpha}{\cos \alpha} = \frac{1}{2} (\cos 30^\circ - \cos(90^\circ - 2\alpha))$$

$$\frac{2 \sin^3 \alpha}{\cos \alpha} = \frac{\sqrt{3}}{2} - \sin 2\alpha \Rightarrow \frac{2 \sin^3 \alpha}{\cos \alpha} + 2 \sin \alpha \cos \alpha = \frac{\sqrt{3}}{2}$$

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$$2\sin^3\alpha + 2\sin\alpha \cdot \cos^2\alpha = \frac{\sqrt{3}}{2}\cos\alpha \Rightarrow 2\sin\alpha(\sin^2\alpha + \cos^2\alpha) = \frac{\sqrt{3}}{2}\cos\alpha$$

$$\tan\alpha = \frac{\sqrt{3}}{4} \Rightarrow \cos\alpha = \frac{4\sqrt{19}}{19}$$

The 2) $\frac{AB}{AC} = \frac{1}{\sqrt{3}}$ second case is found in the same way.