

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$, let D be a point in its interior such that: $\angle BAD = \angle DBC = \angle DCA$, $\angle ADC = 90^\circ$ and $AB \cdot AC = \frac{\sqrt{3}}{4} BC^2$.

Find $\cos(\angle BAD)$.

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Given that $\angle BAD = \angle DBC = \angle DCA$

Suppose, $\angle BAD = \angle DBC = \angle DCA = \alpha$, also $\angle DAC = 90^\circ$ and $AB \cdot AC = \frac{\sqrt{3}}{4} BC^2$

[Assume that, $BC = a, AC = b$ and $AB = c$] as, $\angle DAC = 90^\circ$, So from $\triangle ADC$ we get $\angle DCA = 90^\circ - \alpha$. and also from $\triangle BAC$, we get, $\angle BAC = \alpha + 90^\circ - \alpha = 90^\circ$

So, $\triangle BAC$ is a right angled triangle

$$AB \cdot AC = \frac{\sqrt{3}}{4} BC^2 \Rightarrow bc = \frac{\sqrt{3}}{4} a^2 \Rightarrow (a \sin B)(a \cos B) = \frac{\sqrt{3}}{4} a^2$$

$$[\text{since, } \angle BAC = 90^\circ, \frac{b}{a} = \sin B, \frac{c}{a} = \cos B] \Rightarrow \frac{a^2}{2} [2 \sin B \cos B] = \frac{\sqrt{3}}{4} a^2 \Rightarrow \sin 2B = \frac{\sqrt{3}}{2}$$

So, $\angle B = 30^\circ$ or 60° , Hence, $\angle ABD = \angle B - \alpha, AD = b \sin \alpha$

From $\triangle ABD$, we get: $\angle ADB = 180^\circ - \alpha - \angle B - \alpha = 180^\circ - \angle B$

$$\frac{AB}{\sin \angle ADB} = \frac{AD}{\sin \angle ABD} \Rightarrow \frac{c}{\sin B} = \frac{b \sin \alpha}{\sin(B - \alpha)} \Rightarrow \frac{b}{c} \sin B \sin \alpha = \sin(B - \alpha)$$

$$\Rightarrow \tan B \sin B \sin \alpha = \sin B \cos \alpha - \cos B \sin \alpha$$

$$\Rightarrow \tan B \sin \alpha = \cos \alpha - \cot B \sin \alpha \Rightarrow \sin \alpha (\tan B + \cot B) = \cos \alpha$$

$$\Rightarrow \tan \alpha = \frac{1}{\tan B + \cot B} = \sin B \cos B = \frac{\sqrt{3}}{4}$$

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} = \frac{1}{\sqrt{1 + \frac{3}{16}}} = \frac{1}{\sqrt{\frac{19}{16}}} = \frac{4\sqrt{19}}{19}$$