ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c and λ be positive real numbers satisfying $a+b+c=3$.

Prove that the following inequality holds:

$$\frac{a^5}{\lambda a+b+1}+\frac{b^5}{\lambda b+c+1}+\frac{c^5}{\lambda c+a+1}\ge\frac{3}{\lambda+2}$$

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Lemma: $\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\ge 3$

$$\textit{Proof}: \frac{\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}}{3}\overset{\text{Power Mean}}{\underbrace{\ge}}\left(\frac{a+b+c}{3}\right)^{\frac{5}{2}}=\left(\frac{3}{3}\right)^{\frac{5}{2}}=1\Leftrightarrow$$

$$\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\ge 3$$

$$\frac{a^5}{\lambda a+b+1}+\frac{\lambda a+b+1}{(\lambda+2)^2}\overset{AM-GM}{\underbrace{\ge}}2\sqrt{\frac{a^5}{\lambda a+b+1}\cdot\frac{\lambda a+b+1}{(\lambda+2)^2}}=\frac{2\sqrt{a^5}}{\lambda+2}$$

$$\therefore\sum_{cyc}\frac{a^5}{\lambda a+b+1}+\sum_{cyc}\frac{\lambda a+b+1}{(\lambda+2)^2}\ge\sum_{cyc}\frac{2\sqrt{a^5}}{\lambda+2}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^5}{\lambda a+b+1}+\frac{\lambda(a+b+c)+(a+b+c)+3}{(\lambda+2)^2}\ge\frac{2\left(\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\right)}{\lambda+2}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^5}{\lambda a+b+1}+\frac{\lambda(3)+(3)+3}{(\lambda+2)^2}\ge\frac{2\left(\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\right)}{\lambda+2}\overset{\text{Lemma}}{\underbrace{\ge}}\frac{2(3)}{\lambda+2}=\frac{6}{\lambda+2}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^5}{\lambda a+b+1}+\frac{3(\lambda+2)}{(\lambda+2)^2}\ge\frac{6}{\lambda+2}\Leftrightarrow\sum_{cyc}\frac{a^5}{\lambda a+b+1}+\frac{3}{\lambda+2}\ge\frac{6}{\lambda+2}$$

$$\frac{a^5}{\lambda a+b+1}+\frac{b^5}{\lambda b+c+1}+\frac{c^5}{\lambda c+a+1}\ge\frac{3}{\lambda+2}$$

Equality holds iff $a=b=c=1$.