ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be positive real numbers such that $a+b+c=3M$.

Prove that the following inequality holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\sqrt{\frac{a+1}{2(b+c+2)}}+\sqrt{\frac{b+1}{2(c+a+2)}}+\sqrt{\frac{c+1}{2(a+b+2)}}\geq 3$$

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Solution by Jenish Rijal-Bardiya-Nepal

Here, $$\sqrt{\frac{a+1}{2(b+c+2)}}=\frac{a+1}{\sqrt{2(a+1)(b+c+2)}}\overset{AM-GM}{\geq}\frac{2(a+1)}{2a+b+c+4}$$

Let $S:=a+b+c=3M$. *Thus it suffices to prove that:*

$$\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}\right)\geq 3\Leftrightarrow\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}-1\right)\geq 0$$

Notice that: $$\frac{a}{S-a}+\frac{2a+2}{S+a+4}-1=\frac{(S+2)(2a-b-c)}{(S-a)(S+a+4)}$$

$$\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}-1\right)=\sum_{cyc}\frac{(S+2)(2a-b-c)}{(S-a)(S+a+4)}=$$

$$=\sum_{cyc}(a-b)(S+2)\left(\frac{1}{(S-a)(S+a+4)}-\frac{1}{(S-b)(S+b+4)}\right)$$

$$=\sum_{cyc}(a-b)(S+2)\left[\frac{(S-b)(S+b+4)-(S-a)(S+a+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\right]$$

$$=\sum_{cyc}(a-b)(S+2)\left[\frac{(a-b)(a+b+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\right]=$$

$$=\sum_{cyc}\frac{(a-b)^2(S+2)(a+b+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\geq 0$$

Equality holds if and only if $a=b=c=M$.