ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 1$, *prove that*:

$$\left(\log_{1+a^3}\left(b^2+b^{b^2}\right)\right)^3+\left(\log_{1+b^3}\left(c^2+c^{c^2}\right)\right)^3+\left(\log_{1+c^3}\left(a^2+a^{a^2}\right)\right)^3 \geq$$

$$\geq \left(\log_{1+a^3}\left(b^2+b^{b^2}\right)\right)^2+\left(\log_{1+b^3}\left(c^2+c^{c^2}\right)\right)^2+\left(\log_{1+c^3}\left(a^2+a^{a^2}\right)\right)^2$$

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Lemma: $t^{t^2}+t^2 \geq t^3+1 \quad \forall\, t \geq 1$

$$t^{t^2}=\left(1+(t-1)\right)^{t^2} \overbrace{\geq}^{Bernoulli} 1+t^2(t-1) \Leftrightarrow t^{t^2}+t^2 \geq t^3+1 \quad \forall\, t \geq 1$$

Let $x=\log_{1+a^3}\left(b^2+b^{b^2}\right) \overbrace{\geq}^{Lemma} \log_{1+a^3}(b^3+1)>0$,

$$y=\log_{1+b^3}\left(c^2+c^{c^2}\right) \geq \log_{1+b^3}(c^3+1)>0 \text{ and}$$

$$z=\log_{1+c^3}\left(a^2+a^{a^2}\right) \geq \log_{1+c^3}(a^3+1)>0$$

We have:

$$xyz \geq \log_{1+a^3}(b^3+1)\cdot \log_{1+b^3}(c^3+1)\cdot \log_{1+c^3}(a^3+1)=$$

$$=\frac{\ln(b^3+1)}{\ln(1+a^3)}\cdot\frac{\ln(c^3+1)}{\ln(1+b^3)}\cdot\frac{\ln(a^3+1)}{\ln(1+c^3)}=1$$

Thus, it suffices to prove that: $x^3+y^3+z^3 \geq x^2+y^2+z^2$ with $xyz \geq 1$

WLOG assume that: $x \geq y \geq z \Rightarrow x^2 \geq y^2 \geq z^2$

$$x^3+y^3+z^3 \overbrace{\geq}^{Chebyshev} \frac{1}{3}(x+y+z)(x^2+y^2+z^2) \overbrace{\geq}^{AM-GM}$$

$$\geq \sqrt[3]{xyz}(x^2+y^2+z^2) \overbrace{\geq}^{xyz\geq 1} x^2+y^2+z^2$$

Thus $x^3+y^3+z^3 \geq x^2+y^2+z^2$.

Equality holds if and only if $a=b=c=1$.