ROMANIAN MATHEMATICAL MAGAZINE

$Let\ a, b, c\ be\ non-negative\ real\ numbers\ such\ that\ \ a + b + c = 3.$

$Find\ the\ smallest\ value\ of\ K\ so\ that: K(1 + abc) \geq a^2b + b^2c + c^2a.$

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$$Take\ a = 2, b = 1\ and\ c = 0. So\ our\ inequality\ becomes:$$

$$K(1 + 2 \cdot 1 \cdot 0) \geq 2^2 \cdot 1 + 1^2 \cdot 0 + 0^2 \cdot 2 \implies K \geq 4$$

$Now, let's\ prove\ that\ the\ smallest\ value\ of\ K\ is\ 4.$

$$So\ we\ have\ to\ prove\ that:\ 4(1 + abc) \geq a^2b + b^2c + c^2a$$

$Since\ the\ inequality\ is\ invariant\ under\ cyclic\ permutation, we\ may\ assume$

$W.L.O.G. that\ b\ is\ the\ median(mid - value)\ of\ a, b, and\ c\ (i.e., b\ is\ between\ a\ and\ c).$

$We\ have: a^2b + b^2c + c^2a + abc = b(a + c)^2 - c(a - b)(b - c)$

$$Notice\ that: (a - b)(b - c) \geq 0 \implies c(a - b)(b - c) \geq 0$$

$$\therefore\ a^2b + b^2c + c^2a + abc = b(a + c)^2 - c(a - b)(b - c) \leq b(a + c)^2$$

$$b(a + c)^2 = 4 \cdot b\left(\frac{a + c}{2}\right)\left(\frac{a + c}{2}\right) \overset{AM-GM}{\leq} 4\left(\frac{b + \frac{a + c}{2} + \frac{a + c}{2}}{3}\right)^3 =$$

$$= 4\left(\frac{a + b + c}{3}\right)^3 = 4\left(\frac{3}{3}\right)^3 = 4$$

$$\therefore\ a^2b + b^2c + c^2a + abc \leq b(a + c)^2 \leq 4$$

$$\Leftrightarrow a^2b + b^2c + c^2a \leq 4 - abc \overset{0 \leq 5abc}{\leq} 4 - abc + 5abc = 4(1 + abc)$$

$$Hence, the\ smallest\ value\ of\ K\ is\ 4.$$