ROMANIAN MATHEMATICAL MAGAZINE

Let $a_1,\ a_2, \dots\dots, a_n (n \geq 3)$ be positive real numbers such that

$$\sum_{i=1}^{n} a_i^3 \leq n. \textit{ Prove that:}$$

$$\sum_{i=1}^{n} \sqrt{7a_i^2 - 12a_i + 9} \geq 2\sum_{i=1}^{n} a_i^2$$

Proposed by Nguyen Van Hoa-Vietnam

Solution by Jenish Rijal-Nepal

Claim: $\sqrt{7a^2 - 12a + 9} \geq 2a^2 - \frac{7}{6}(a^3 - 1)$

Proof: $\sqrt{7a^2 - 12a + 9} - 2a^2 + \frac{7}{6}(a^3 - 1) =$

$$= \left(\sqrt{7a^2 - 12a + 9} - \frac{a+3}{2}\right) + \left(\frac{a+3}{2} - 2a^2 + \frac{7}{6}(a^3 - 1)\right)$$

$$\sqrt{7a^2 - 12a + 9} - \frac{a+3}{2} = \frac{(7a^2 - 12a + 9) - \frac{(a+3)^2}{4}}{\sqrt{7a^2 - 12a + 9} + \frac{a+3}{2}} =$$

$$= \frac{27(a-1)^2}{4\left(\sqrt{7a^2 - 12a + 9} + \frac{a+3}{2}\right)}$$

$$\frac{a+3}{2} - 2a^2 + \frac{7}{6}(a^3 - 1) = \frac{7}{6}a^3 - 2a^2 + \frac{1}{2}a + \frac{1}{3} = \frac{(a-1)^2(7a+2)}{6}$$

$$\sqrt{7a^2 - 12a + 9} - 2a^2 + \frac{7}{6}(a^3 - 1) = (a-1)^2\left[\frac{27}{4\left(\sqrt{7a^2 - 12a + 9} + \frac{a+3}{2}\right)} + \frac{(7a+2)}{6}\right] \geq 0$$

$$\therefore \sqrt{7a^2 - 12a + 9} \geq 2a^2 - \frac{7}{6}(a^3 - 1)$$

$$\Rightarrow \sum_{i=1}^{n} \sqrt{7a_i^2 - 12a_i + 9} \geq 2\sum_{i=1}^{n} a_i^2 - \frac{7}{6}\left(\sum_{i=1}^{n} a_i^3 - n\right)$$

Since $\sum_{i=1}^{n} a_i^3 - n \leq 0$, *we have* $-\frac{7}{6}\left(\sum_{i=1}^{n} a_i^3 - n\right) \geq 0$:

$$\sum_{i=1}^{n} \sqrt{7a_i^2 - 12a_i + 9} \geq 2\sum_{i=1}^{n} a_i^2 - \frac{7}{6}\left(\sum_{i=1}^{n} a_i^3 - n\right) \geq 2\sum_{i=1}^{n} a_i^2 - 0 = 2\sum_{i=1}^{n} a_i^2$$

$$\textit{Therefore}: \sum_{i=1}^{n} \sqrt{7a_i^2 - 12a_i + 9} \geq 2\sum_{i=1}^{n} a_i^2$$

Equality holds if and only if $a_1 = a_2 = \dots\dots = a_n = 1$.