ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ and $\frac{1}{a}+\frac{1}{b}=2$ then prove that :

$$(1+a)^{\frac{1-a}{a}}\cdot(1+b)^{\frac{1-b}{b}}\leq 1$$

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Let $\frac{1}{a}=x$ & $\frac{1}{b}=y$ and let $f(t)=(t-1).\ln\left(1+\frac{1}{t}\right)\ \forall\ t\in(0,2)$ and then :

$f''(t)=-\frac{3t+1}{t^2(t^2+1)^2}<0\ (\because 0<t<2)\Rightarrow f(t)$ is concave $\therefore f(x)+f(y)\overset{\text{Jensen}}{\leq}$

$$2\left(\frac{x+y}{2}-1\right).\ln\left(1+\frac{1}{\frac{x+y}{2}}\right)\ (\because x,y>0 \text{ and } \because x+y=2\Rightarrow x,y<2)$$

$$\overset{x+y=2}{=}2(1-1).\ln\left(1+\frac{1}{1}\right)=0\therefore\left(\frac{1}{a}-1\right).\ln(1+a)+\left(\frac{1}{b}-1\right).\ln(1+b)\leq 0$$

$$\Rightarrow\ln\left((1+a)^{\frac{1-a}{a}}.(1+b)^{\frac{1-b}{b}}\right)\leq 0\Rightarrow(1+a)^{\frac{1-a}{a}}.(1+b)^{\frac{1-b}{b}}\leq 1$$

$$\forall\ a,b>0\ \Big|\ \frac{1}{a}+\frac{1}{b}=2,''=''\text{ iff } a=b=1\ (\text{QED})$$