ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ and $\frac{1}{a}+\frac{1}{b}=2$ then prove that :

$$(1+a^2)^{\frac{1-a}{a}}.(1+b^2)^{\frac{1-b}{b}}\leq 1$$

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$$\text{Let } \frac{1}{a}=x \text{ and } \frac{1}{b}=y \text{ and let } f(t)=(t-1).\ln\left(1+\frac{1}{t^2}\right)\ \forall\, t\in(0,2)$$

$$\text{and then : } f''(t)=\frac{2(t^3-3t^2-t-1)}{t^2(t^2+1)^2}=\frac{2(t^2(t-2)-t^2-t-1)}{t^2(t^2+1)^2}<0$$

$$(\because 0<t<2)\Rightarrow f(t) \text{ is concave } \therefore f(x)+f(y)\overset{\text{Jensen}}{\leq}$$

$$2\left(\frac{x+y}{2}-1\right).\ln\left(1+\frac{1}{\left(\frac{x+y}{2}\right)^2}\right)\ (\because x,y>0 \text{ and } \because x+y=2\Rightarrow x,y<2)$$

$$\overset{x+y=2}{=}2(1-1).\ln\left(1+\frac{1}{1}\right)=0\ \therefore\left(\frac{1}{a}-1\right).\ln(1+a^2)+\left(\frac{1}{b}-1\right).\ln(1+b^2)\leq 0$$

$$\Rightarrow \ln\left((1+a^2)^{\frac{1-a}{a}}.(1+b^2)^{\frac{1-b}{b}}\right)\leq 0\Rightarrow (1+a^2)^{\frac{1-a}{a}}.(1+b^2)^{\frac{1-b}{b}}\leq 1$$

$$\forall\, a,b>0\ \Big|\ \frac{1}{a}+\frac{1}{b}=2,''='' \text{ iff } a=b=1 \text{ (QED)}$$