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If $a,b,c\geq 0$ and $ab+bc+ca>0$ then prove that :

$$\sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c}\geq\sum_{cyc}a$$

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When exactly one variable equals to zero & WLOG we may assume $a=0$ $(b,c>0)$, then it remains to prove : $b.\sqrt{\frac{b}{c}}+c.\sqrt{\frac{c}{b}}\overset{?}{\geq}b+c$

$$\Leftrightarrow\frac{(\sqrt{b}-\sqrt{c})(b.\sqrt{b}-c.\sqrt{c})}{\sqrt{bc}}\overset{?}{\geq}0\Leftrightarrow\frac{(\sqrt{b}-\sqrt{c})^2.(b+c+\sqrt{bc})}{\sqrt{bc}}\overset{?}{\geq}0\rightarrow\text{true}$$

$$\therefore\sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c}\geq\sum_{cyc}a$$ and we now focus on the case when : $a,b,c>0$

and we assign : $m=ab+ca, n=bc+ab, p=ca+bc$ and then :

$$\sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c}=\sum_{cyc}\frac{a^2}{\frac{m}{\sqrt{n+p}}}\overset{\text{Bergstrom}}{\geq}\frac{(\sum_{cyc}a)^2}{\sum_{cyc}\frac{m}{\sqrt{n+p}}}\overset{\text{CBS}}{\geq}\frac{(\sum_{cyc}a)^2}{\sqrt{\sum_{cyc}m}\sqrt{\sum_{cyc}\frac{m}{n+p}}}$$

$$\therefore\sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c}\geq\frac{(\sum_{cyc}a)^2}{\sqrt{2\sum_{cyc}ab}.\sqrt{\frac{\sum_{cyc}m^3+(2\sum_{cyc}ab)(\sum_{cyc}mn)}{(2\sum_{cyc}ab)(\sum_{cyc}mn)-mnp}}}\rightarrow(1)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $abc=r^2s,\sum_{cyc}a=s, abc=r^2s,\sum_{cyc}ab=4Rr+r^2,\sum_{cyc}a^2=s^2-8Rr-2r^2,$

$$\sum_{cyc}a^3=s^3-12Rrs,\sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2),$$

$$\sum_{cyc}a^3b^3=(4Rr+r^2)^3-12Rr^3s^2$$ and then : $$\sum_{cyc}m^3=\sum_{cyc}\left(\sum_{cyc}ab-bc\right)^3$$

$$=3\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)-\sum_{cyc}a^3b^3\therefore\sum_{cyc}m^3=$$

$3r^2(4Rr+r^2)((4R+r)^2-2s^2)-(4Rr+r^2)^3+12Rr^3s^2\rightarrow(2)$ and also,

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$$\sum_{cyc} mn = \sum_{cyc}\left(bc\left(a^2+\sum_{cyc} ab\right)\right) \therefore \sum_{cyc} mn = r^2s^2+(4Rr+r^2)^2 \to (3)$$

$$\therefore (1),(2) \text{ and } (3) \Rightarrow \sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c} \geq$$

$$\frac{s^2}{\sqrt{2(4Rr+r^2)}.\sqrt{\frac{3r^2(4Rr+r^2)((4R+r)^2-2s^2)-(4Rr+r^2)^3+12Rr^3s^2+2(4Rr+r^2)\left(r^2s^2+(4Rr+r^2)^2\right)}{2(4Rr+r^2)\left(r^2s^2+(4Rr+r^2)^2\right)-r^2s.4Rrs}}} \overset{?}{\geq} \sum_{cyc} a$$

$$= s \Leftrightarrow (2R+r)s^4+(64R^3+64R^2r+32Rr^2+5r^3)s^2-4r(4R+r)^4 \overset{?}{\geq} 0 \text{ and } \because$$

$$(2R+r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{ suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} (2R+r)(s^2-16Rr+5r^2)^2 \Leftrightarrow$$

$$(64R^3+128R^2r+44Rr^2-5r^3)s^2 \overset{?}{\geq} r\begin{pmatrix}1024R^4+1536R^3r+320R^2r^2-\\46Rr^3+29r^4\end{pmatrix}$$

$$\text{and finally, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (64R^3+128R^2r+44Rr^2-5r^3)(16Rr-5r^2)$$

$$\overset{?}{\geq} \text{RHS of } (**) \Leftrightarrow (R-2r)(96R^2+64Rr+r^2) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c} \geq \sum_{cyc} a \text{ and so,}$$

$$\sum_{cyc}\frac{a.\sqrt{ab+ac+2bc}}{b+c} \geq \sum_{cyc} a \; \forall a,b,c \geq 0 \text{ and } ab+bc+ca > 0,$$

$$'' = '' \text{ iff } \begin{pmatrix}a=0\\b=c>0\end{pmatrix} \text{ or } \begin{pmatrix}b=0\\c=a>0\end{pmatrix} \text{ or } \begin{pmatrix}c=0\\a=b>0\end{pmatrix} \text{ or } (a=b=c>0) \text{ (QED)}$$