ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca+abc\le 4$ then prove that :

$$a^2+b^2+c^2+a+b+c\ge 2(ab+bc+ca)$$

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$$4\ge\sum_{cyc}ab+abc\overset{AM-GM}{\ge}4.\sqrt[4]{(abc)^3}\Rightarrow abc\le 1\Rightarrow x^3y^3z^3\le 1$$

$$\left(x=\sqrt[3]{a},y=\sqrt[3]{b},z=\sqrt[3]{c}\right)\Rightarrow xyz\le 1\rightarrow (1)$$

$$\text{Now},\sum_{cyc}a^2+\sum_{cyc}a-2\sum_{cyc}ab=\sum_{cyc}x^6+\sum_{cyc}x^3-2\sum_{cyc}x^3y^3$$

$$\overset{via\ (1)}{\ge}\sum_{cyc}x^6+xyz\sum_{cyc}x^3-2\sum_{cyc}x^3y^3$$

$$\overset{Schur}{\ge}\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4-3x^2y^2z^2+xyz\sum_{cyc}x^3-2\sum_{cyc}x^3y^3$$

$$\overset{AM-GM}{\ge}2\sum_{cyc}x^3y^3-3x^2y^2z^2+xyz\sum_{cyc}x^3-2\sum_{cyc}x^3y^3=xyz\left(\sum_{cyc}x^3-3xyz\right)$$

$$\overset{AM-GM}{\ge}0\ \text{and so},a^2+b^2+c^2+a+b+c\ge 2(ab+bc+ca)$$

$\forall\ a,b,c>0$ and $ab+bc+ca+abc\le 4,''='' $ iff $a=b=c=1$ (QED)