ROMANIAN MATHEMATICAL MAGAZINE

If $a^3+b^3+c^3\geq 4+3abc$ then prove that :

$$a+b+c+\sqrt{2}.(|a-b|+|b-c|+|c-a|)\geq 6$$

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$$\sum_{cyc}a^3\geq 4+3abc\Rightarrow 3abc+\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\geq 4+3abc$$

$$\Rightarrow\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\geq 4\rightarrow (1)\ and\ \because 4>0\ and\ \sum_{cyc}a^2-\sum_{cyc}ab=$$

$$\frac{1}{2}\sum_{cyc}(a-b)^2\geq 0\ \forall\ a,b,c\in\mathbb{R}\therefore (1)\Rightarrow\sum_{cyc}a>0\rightarrow (2)\ and$$

$$\sqrt{\sum_{cyc}a^2-\sum_{cyc}ab}\geq\frac{2}{\sqrt{\sum_{cyc}a}}\rightarrow (3)\ and\ now,\sum_{cyc}a+\sqrt{2}.\sum_{cyc}|a-b|=$$

$$\sum_{cyc}a+\sqrt{2}.\sqrt{\sum_{cyc}(a-b)^2+2\sum_{cyc}(|a-b||c-a|)}$$

$$\overset{Triangle\ Inequality}{\geq}\sum_{cyc}a+\sqrt{2}.\sqrt{2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+2\left|\sum_{cyc}((a-b)(c-a))\right|}$$

$$=\sum_{cyc}a+\sqrt{2}.\sqrt{2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+2\left|\sum_{cyc}ab-\sum_{cyc}a^2-\sum_{cyc}ab+\sum_{cyc}ab\right|}$$

$$=\sum_{cyc}a+\sqrt{2}.\sqrt{2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}$$

$$\left(\because\sum_{cyc}a^2-\sum_{cyc}ab=\frac{1}{2}\sum_{cyc}(a-b)^2\geq 0\ \forall\ a,b,c\in\mathbb{R}\right)$$

$$=\sum_{cyc}a+2\sqrt{2}.\sqrt{\sum_{cyc}a^2-\sum_{cyc}ab}\overset{via\ (3)}{\geq}\sum_{cyc}a+\frac{4.\sqrt{2}}{\sqrt{\sum_{cyc}a}}$$

$$=\sum_{cyc}a+\frac{2\sqrt{2}}{\sqrt{\sum_{cyc}a}}+\frac{2\sqrt{2}}{\sqrt{\sum_{cyc}a}}\overset{AM-GM}{\geq}3.\sqrt[2]{\frac{8\left(\sum_{cyc}a\right)}{\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}a}}}\left(\because\sum_{cyc}a>0,via\ (2)\right)$$

$$=6\ and\ so,a+b+c+\sqrt{2}.(|a-b|+|b-c|+|c-a|)\geq 6$$

$$\forall\ a,b,c\in\mathbb{R}\ \Big|\ \sum_{cyc}a^3\geq 4+3abc\ (QED)$$