ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ and $a+b+ab=3$ then prove that :

$$(1+a)^{1-\sqrt{b}}(1+b)^{1-\sqrt{a}}\geq 1$$

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$$a+b+ab=3\Rightarrow \boxed{b=\frac{3-a}{1+a}}\therefore b\geqslant 1\Rightarrow 3-a\geqslant 1+a\Rightarrow a\leqslant 1$$

$$\therefore b\geqslant 1\Rightarrow a\leqslant 1\rightarrow (*)$$

$$\boxed{\text{WLOG we may assume : } b\leq 1}\ (\text{then, consequently } ''a'' \text{ will be } \geq 1;\text{via } (*))$$

$$\text{and then : } (1+a)^{1-\sqrt{b}}.(1+b)^{1-\sqrt{a}}\overset{?}{\geq}1$$

$$\Leftrightarrow \left(\frac{1}{1+a}\right)^{1-\sqrt{b}}.\left(\frac{1}{1+b}\right)^{1-\sqrt{a}}\overset{?}{\geq}1 \text{ and now, } \frac{1}{1+a},\frac{1}{1+b}=1-\frac{a}{1+a},1-\frac{b}{1+b}$$

$$\text{and } \because -\frac{a}{1+a},-\frac{b}{1+b}>-1 \text{ and } \because 1-\sqrt{b},1-\sqrt{a}<1$$

$$\therefore \left(\frac{1}{1+a}\right)^{1-\sqrt{b}}.\left(\frac{1}{1+b}\right)^{1-\sqrt{a}}\overset{\text{Bernoulli}}{\leq}\left(\frac{1}{1+a}\right)^{1-\sqrt{b}}.\left(1-\frac{b}{1+b}.(1-\sqrt{a})\right)$$

$$\overset{\text{Bernoulli}}{\leq}\left(1-\frac{a}{1+a}.(1-\sqrt{b})\right).\left(1-\frac{b}{1+b}.(1-\sqrt{a})\right)$$

$$\left(\because 1-\frac{b}{1+b}.(1-\sqrt{a})=1+\frac{b}{1+b}.(\sqrt{a}-1)>0 \text{ as } a\geq 1\right)$$

$$=1+\frac{ab(1-\sqrt{a})(1-\sqrt{b})}{(1+a)(1+b)}+\frac{b(\sqrt{a}-1)}{1+b}+\frac{a(\sqrt{b}-1)}{1+a}\leq$$

$$1+\frac{b(\sqrt{a}-1)}{1+b}-\frac{a(1-\sqrt{b})}{1+a}\ (\because b\leq 1\leq a\Rightarrow (1-\sqrt{a})(1-\sqrt{b})\leq 0)\overset{?}{\leq}1$$

$$\Leftrightarrow \frac{b(\sqrt{a}-1)}{1+b}\overset{?}{\leq}\frac{a(1-\sqrt{b})}{1+a}\Leftrightarrow \frac{\left(\frac{3-a}{1+a}\right)(\sqrt{a}-1)}{1+\frac{3-a}{1+a}}\overset{?}{\leq}\frac{a\left(1-\sqrt{\frac{3-a}{1+a}}\right)}{1+a}$$

$$\Leftrightarrow 4a.\sqrt{\frac{3-a}{1+a}}\overset{?}{\geq}4a-(3-a)(\sqrt{a}-1)(1+a)$$

$$\text{Now, } 4a-4a.\sqrt{\frac{3-a}{1+a}}=4a.\frac{1-\frac{3-a}{1+a}}{1+\sqrt{\frac{3-a}{1+a}}}=\frac{4a.2(\sqrt{a}-1)(\sqrt{a}+1)}{(1+a)\left(1+\sqrt{\frac{3-a}{1+a}}.1\right)}$$

$$\overset{AM-GM}{\geq} \frac{8a\left(\sqrt{a}-1\right)\left(\sqrt{a}+1\right)}{(1+a)\left(1+\frac{2}{1+a}\right)} = \frac{8a\left(\sqrt{a}-1\right)\left(\sqrt{a}+1\right)}{3+a} \overset{?}{\geq} (3-a)\left(\sqrt{a}-1\right)(1+a)$$

$$\Leftrightarrow \left(\sqrt{a}-1\right)\left(8a\left(\sqrt{a}+1\right)-(9-a^2)(1+a)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\sqrt{a}-1\right)^2\left(a^{\frac{5}{2}}+a^2+2a^{\frac{3}{2}}+10a+9\right) \overset{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore (1+a)^{1-\sqrt{b}}.(1+b)^{1-\sqrt{a}} \geq 1 \; \forall \, a,b>0 \,\big|\, a+b+ab=3,$$
$$'' = '' \text{ iff } a=b=1 \text{ (QED)}$$