ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c>0, a+b+c=2\ then:$$

$$\sqrt{2}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)-\sqrt{11+\frac{4}{abc}}\geq\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x^2=a, y^2=b, z^2=c\ then\ x^2+y^2+z^2=2$$

$$Let\ p=xyz, q=xy+yz+zx, r=x+y+z$$

$$x^2+y^2+z^2=2\ or, 3(xyz)^{\frac{2}{3}}\overset{AM-GM}{\leq}2\ or, xyz\leq\left(\frac{2}{3}\right)^{\frac{3}{2}}\ or, p\leq\left(\frac{2}{3}\right)^{\frac{3}{2}}(1)$$

$$r^2=(x+y+z)^2=\sum x^2+2\sum xy=2+2q\ (2)\ \&\ r^2=(x+y+z)^2\leq 3\sum x^2=6\ (3)$$

$$q=xy+yz+zx\overset{AM-GM}{\geq}3(xyz)^{\frac{2}{3}}=3r^{\frac{2}{3}}\ then\ 4q\geq 12p^{\frac{2}{3}}(4)$$

$$\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)=\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{r}{p}\ \&\ \sqrt{11+\frac{4}{abc}}=\sqrt{11+\frac{4}{p^2}}$$

$$We\ need\ to\ show:$$

$$\sqrt{2}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)-\sqrt{11+\frac{4}{abc}}\geq\sqrt{2}$$

$$or, \frac{\sqrt{2}r}{p}-\sqrt{11+\frac{4}{p^2}}\geq\sqrt{2}\ or, \sqrt{2}(r-p)\geq\sqrt{4+11p^2}$$

$$2(r-p)^2\geq 4+11p^2$$

$$or, 2r^2-4rp-9p^2\geq 4\ or, 2(2+2q)-4rp-9p^2\overset{(2)}{\geq}4$$

$$4q \geq 4rp + 9p^2$$

$$12p^{\frac{2}{3}} \overset{(4)\ \&\ (3)}{\geq} 4\sqrt{6}p + 9p^2$$

$$12 \geq 4\sqrt{6}p^{\frac{1}{3}} + 9p^{\frac{4}{3}}\ true$$

$$\left(as\ 4\sqrt{6}p^{\frac{1}{3}} + 9p^{\frac{4}{3}} \overset{(1)}{\leq} 4\sqrt{6} \times \left(\frac{2}{3}\right)^{\frac{1}{2}} + 9\left(\frac{2}{3}\right)^{2} = 4 \times 2 + 4 = 12\right)$$

$$Equality\ holds\ for\ a = b = c = \frac{2}{3}$$