

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{3}{2}$ then prove that :
 $(3a+1)(3b+1)(3c+1) \geq 64$

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Let $1+a=x, 1+b=y, 1+c=z$ and then : $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{3}{2}$

$$\Rightarrow \sum_{\text{cyc}} \frac{1}{x} = \frac{3}{2} \Rightarrow 2 \sum_{\text{cyc}} xy = 3xyz \rightarrow (*)$$

Now, $(3a+1)(3b+1)(3c+1) \stackrel{?}{\geq} 64 \Leftrightarrow (3x-2)(3y-2)(3z-2) \stackrel{?}{\geq} 64$

$$\Leftrightarrow 9xyz - 6 \sum_{\text{cyc}} xy + 4 \sum_{\text{cyc}} x - 24 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 6 \sum_{\text{cyc}} xy - 6 \sum_{\text{cyc}} xy + 4 \sum_{\text{cyc}} x - 24 \cdot \frac{3xyz}{2 \sum_{\text{cyc}} xy} \stackrel{?}{\geq} 0 \text{ (via (*))}$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} x \right) \left(\sum_{\text{cyc}} xy \right) \stackrel{?}{\geq} 9xyz \rightarrow \text{true} \because$$

$$\sum_{\text{cyc}} x \stackrel{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{xyz} \text{ and } \sum_{\text{cyc}} xy \stackrel{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{x^2 y^2 z^2}$$

$\therefore (3a+1)(3b+1)(3c+1) \geq 64 \forall a, b, c > 0 \mid \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{3}{2}$,
 " = " iff $a = b = c = 1$ (QED)