

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $3 + a + b + c = ab + bc + ca + 3abc$ then prove that :

$$\frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} \leq \frac{3}{4}$$

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Let $1 + a = x, 1 + b = y, 1 + c = z$ and then : $3 + \sum_{\text{cyc}} a = \sum_{\text{cyc}} ab + 3abc \Rightarrow$

$$\sum_{\text{cyc}} x = \sum_{\text{cyc}} (x(y-1)(z-1)) = 3xyz + \sum_{\text{cyc}} x - 2 \sum_{\text{cyc}} xy \Rightarrow 2 \sum_{\text{cyc}} xy = 3xyz \rightarrow (*)$$

Now, $\sum_{\text{cyc}} \frac{1}{3+a^2} \stackrel{\text{AM-GM}}{\leq} \sum_{\text{cyc}} \frac{1}{2(1+a)} = \frac{1}{2} \cdot \sum_{\text{cyc}} \frac{1}{x} = \frac{1}{2xyz} \cdot \sum_{\text{cyc}} xy \stackrel{\text{via } (*)}{=} \frac{3}{4}$ and so,

$$\frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} \leq \frac{3}{4} \quad \forall a, b, c > 0 \mid 3 + a + b + c = ab + bc + ca + 3abc,$$

" = " iff $a = b = c = 1$ (QED)