

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{3}{2}$ then prove that :
 $ab + bc + ca \geq 3$

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Let $1 + a = x, 1 + b = y, 1 + c = z$ and then : $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{3}{2}$

$$\Rightarrow \sum_{\text{cyc}} \frac{1}{x} = \frac{3}{2} \Rightarrow 2 \sum_{\text{cyc}} xy = 3xyz \rightarrow (*)$$

$$\text{Now, } ab + bc + ca \stackrel{?}{\geq} 3 \Leftrightarrow \sum_{\text{cyc}} ((x-1)(y-1)) \stackrel{?}{\geq} 3 \Leftrightarrow \sum_{\text{cyc}} xy \stackrel{?}{\geq} 2 \sum_{\text{cyc}} x$$

$$\Leftrightarrow \sum_{\text{cyc}} xy \stackrel{?}{\geq} 2 \left(\sum_{\text{cyc}} x \right) \cdot \frac{3xyz}{2 \sum_{\text{cyc}} xy} \text{ (via (*))} \Leftrightarrow \left(\sum_{\text{cyc}} xy \right)^2 \stackrel{?}{\geq} 3xyz \left(\sum_{\text{cyc}} x \right)$$

$$\rightarrow \text{true and so, } ab + bc + ca \geq 3 \forall a, b, c > 0 \mid \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{3}{2},$$

" = " iff $a = b = c = 1$ (QED)