

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $3 + a + b + c = ab + bc + ca + 3abc$ then prove that :

$$\frac{1}{3 + 2a + b} + \frac{1}{3 + 2b + c} + \frac{1}{3 + 2c + a} \leq \frac{1}{2}$$

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Let $1 + a = x, 1 + b = y, 1 + c = z$ and then : $3 + \sum_{cyc} a = \sum_{cyc} ab + 3abc \Rightarrow$

$$\sum_{cyc} x = \sum_{cyc} (x(y-1)(z-1)) = 3xyz + \sum_{cyc} x - 2 \sum_{cyc} xy \Rightarrow 2 \sum_{cyc} xy = 3xyz \rightarrow (*)$$

Now, $\frac{1}{3 + 2a + b} + \frac{1}{3 + 2b + c} + \frac{1}{3 + 2c + a} = \sum_{cyc} \frac{1}{1 + a + 1 + a + 1 + b}$

Reverse Bergstrom $\leq \frac{1}{9} \cdot \sum_{cyc} \left(\frac{2}{1+a} + \frac{1}{1+b} \right) = \frac{1}{3} \cdot \sum_{cyc} \frac{1}{1+a} = \frac{1}{3} \cdot \sum_{cyc} \frac{1}{x} = \frac{1}{3xyz} \cdot \sum_{cyc} xy$

via (*) $\frac{1}{2}$ and so, $\frac{1}{3 + 2a + b} + \frac{1}{3 + 2b + c} + \frac{1}{3 + 2c + a} \leq \frac{1}{2}$

$\forall a, b, c > 0 \mid 3 + a + b + c = ab + bc + ca + 3abc, " = " \text{ iff } a = b = c = 1 \text{ (QED)}$