

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $3 + a + b + c = ab + bc + ca + 3abc$ then prove that :

$$(1 + a)(1 + b)(1 + c) \geq 8$$

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$$\begin{aligned} \text{Let } 1 + a = x, 1 + b = y, 1 + c = z \text{ and then : } 3 + \sum_{\text{cyc}} a &= \sum_{\text{cyc}} ab + 3abc \Rightarrow \\ \sum_{\text{cyc}} x &= \sum_{\text{cyc}} (x(y-1)(z-1)) = 3xyz + \sum_{\text{cyc}} x - 2 \sum_{\text{cyc}} xy \Rightarrow 2 \sum_{\text{cyc}} xy = 3xyz \rightarrow (*) \\ (1 + a)(1 + b)(1 + c) &\stackrel{?}{\geq} 8 \Leftrightarrow xyz \stackrel{?}{\geq} \left(\frac{3xyz}{\sum_{\text{cyc}} xy} \right)^3 \Leftrightarrow \left(\sum_{\text{cyc}} xy \right)^3 \stackrel{?}{\geq} 27x^2y^2z^2 \\ \Leftrightarrow \sum_{\text{cyc}} xy &\stackrel{?}{\geq} 3 \cdot \sqrt[3]{x^2y^2z^2} \rightarrow \text{true via AM - GM and so, } (1 + a)(1 + b)(1 + c) \geq 8 \\ \forall a, b, c > 0 \mid 3 + a + b + c &= ab + bc + ca + 3abc, " = " \text{ iff } a = b = c = 1 \text{ (QED)} \end{aligned}$$