ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $3+a+b+c=ab+bc+ca+3abc$ then prove that :

$$2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right)\leq 3\sqrt{(1+a)(1+b)(1+c)}$$

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$$\text{Let } 1+a=x,1+b=y,1+c=z \text{ and then : } 3+\sum_{cyc}a=\sum_{cyc}ab+3abc\Rightarrow$$

$$\sum_{cyc}x=\sum_{cyc}\left(x(y-1)(z-1)\right)=3xyz+\sum_{cyc}x-2\sum_{cyc}xy\Rightarrow 2\sum_{cyc}xy=3xyz\rightarrow(*)$$

$$\text{Let } \sqrt{x}=u,\sqrt{y}=v \text{ and } \sqrt{z}=w \text{ and then : } (*)\Rightarrow 2\sum_{cyc}u^2v^2=3u^2v^2w^2\rightarrow(**)$$

$$\text{Now, } 2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right)\overset{?}{\leq}3.\sqrt{(1+a)(1+b)(1+c)}$$

$$\Leftrightarrow 2\sum_{cyc}u\overset{?}{\leq}3uvw\Leftrightarrow 2\sum_{cyc}u.\frac{3u^2v^2w^2}{2\sum_{cyc}u^2v^2}\overset{?}{\leq}3uvw\ (\text{via } (**))$$

$$\Leftrightarrow\sum_{cyc}u^2v^2\overset{?}{\geq}uvw.\sum_{cyc}u\rightarrow\text{true}$$

$$\therefore 2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right)\leq 3.\sqrt{(1+a)(1+b)(1+c)}$$

$\forall a,b,c>0\ |\ 3+a+b+c=ab+bc+ca+3abc,''=''$ iff $a=b=c=1$ (QED)