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If $a, b > 0$; $ab = 1$ then:

$$\frac{a}{(b^2 + 1)^3} + \frac{b}{(a^2 + 1)^3} \geq \frac{1}{4}$$

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$$\begin{aligned} \frac{a}{(b^2 + 1)^3} + \frac{b}{(a^2 + 1)^3} &= \frac{a}{(b^2 + ab)^3} + \frac{b}{(a^2 + ab)^3} = \\ &= \frac{a}{b^3(a + b)^3} + \frac{b}{a^3(a + b)^3} = \frac{1}{(a + b)^3} \left(\frac{a}{b^3} + \frac{b}{a^3} \right) = \\ &= \frac{1}{(a + b)^3} \cdot \frac{a^4 + b^4}{(ab)^3} = \frac{a^4 + b^4}{(a + b)^3} \cdot \frac{1}{1^3} \stackrel{\text{BERGSTROM}}{\geq} \frac{(a^2 + b^2)^2}{2(a + b)^3} \geq \frac{\left(\frac{a^2}{1} + \frac{b^2}{1} \right)^2}{2(a + b)^3} \stackrel{\text{BERGSTROM}}{\geq} \\ &\geq \frac{\left(\frac{(a + b)^2}{1 + 1} \right)^2}{2(a + b)^3} = \frac{(a + b)^4}{8(a + b)^3} = \frac{a + b}{8} \stackrel{\text{AM-GM}}{\geq} \frac{2\sqrt{ab}}{8} = \frac{2 \cdot \sqrt{1}}{8} = \frac{1}{4} \end{aligned}$$

Equality holds for $a = b = 1$.