

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $ab = 1$ then:

$$\frac{a}{b^2 + 3a} + \frac{b}{a^2 + 3b} \leq \frac{1}{2}$$

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$$\text{Denote: } \begin{cases} S = a + b \\ P = ab = 1 \end{cases} \stackrel{AM-GM}{\geq} 2\sqrt{ab} = 2\sqrt{1} = 2$$

$$S \geq 2 \Rightarrow S - 2 \geq 0 \quad (1)$$

$$\frac{a}{b^2 + 3a} + \frac{b}{a^2 + 3b} \leq \frac{1}{2}$$

$$2a(a^2 + 2b) + 2b(b^2 + 3a) \leq (b^2 + 3a)(a^2 + 3b)$$

$$2a^3 + 6ab + 2b^3 + 6ab \leq (ab)^2 + 3b^3 + 3a^3 + 9ab$$

$$2(a^3 + b^3) + 6 + 6 \leq 1 + 3(b^3 + a^3) + 9$$

$$a^3 + b^3 + 10 \geq 12 \Leftrightarrow a^3 + b^3 \geq 2 \quad (\text{to prove})$$

$$a^3 + b^3 \geq 2\sqrt{(ab)^3} = 2\sqrt{1^3} = 2 \cdot 1 = 2$$

Equality holds for $a = b = 1$.