ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0; ab = 1$ then:

$$\frac{a}{b^3+1}+\frac{b}{a^3+1}\geq 1$$

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$$\text{Denote: } \begin{cases} S = a+b > 0 \\ P = ab > 0 \end{cases}$$

$$S = a+b \overset{AM-GM}{\geq} 2\sqrt{ab} = 2\cdot\sqrt{1} = 2 \Rightarrow S \geq 2 \Rightarrow S-2 \geq 0 \quad (1)$$

$$\frac{a}{b^3+1}+\frac{b}{a^3+1}\geq 1 \Leftrightarrow a(a^3+1)+b(b^3+1)\geq (a^3+1)(b^3+1)$$

$$a^4+b^4+a+b \geq (ab)^3+a^3+b^3+1$$

$$(S^2-2P)^2-2P^2+S \geq 1+S^3-3SP+1$$

$$(S^2-2)^2-2+S-1-S^3+3S-1 \geq 0$$

$$S^4-4S^2+4-4-S^3+4S \geq 0, \qquad S^4-S^3-4S^2+4S \geq 0$$

$$S^3-S^2-4S+4 \geq 0, \qquad S^2(S-1)-4(S-1) \geq 0$$

$$S^2-4 \geq 0,\ (S-2)(S+2) \geq 0,\ S-2 \geq 0 \quad \text{(True by (1))}$$

Equality holds for $a = b = 1$.