If $a,b>0; a+b=2$ then:

$$\frac{1}{a^2}+\frac{1}{b^2}+6ab\ge 8$$

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Denote: $\begin{cases} S=a+b \\ P=ab \end{cases} \Rightarrow \begin{cases} S=2 \\ a^2+b^2=S^2-2P=4-2P \end{cases}$

$$\frac{1}{a^2}+\frac{1}{b^2}+6ab\ge 8 \Leftrightarrow \frac{a^2+b^2}{a^2b^2}+6ab\ge 8$$

$$\frac{4-2P}{P^2}+6P\ge 8 \Leftrightarrow 4-2P+6P^3-8P^2\ge 0$$

$$6P^3-6P^2-2P^2+2P-4P+4\ge 0,\qquad 6P^2(P-1)-2P(P-1)-4(P-1)\ge 0$$

$$(P-1)(3P^2-P-2)\ge 0,\qquad (P-1)(3P^2-3P+2P-2)\ge 0$$

$$(P-1)[3P(P-1)+2(P-1)]\ge 0,\ \ (P-1)^2(3P+2)\ge 0\quad \text{(true)}$$

$$a,b>0\Rightarrow ab>0\Rightarrow P>0\Rightarrow 3P+2>0$$

Equality holds for $ab=1\Rightarrow a=b=1$.