ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0; a + b = 2$ then:

$$\frac{1}{a^3} + \frac{1}{b^3} + 12ab \geq 14$$

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Denote: $\begin{cases} S = a + b \\ P = ab \end{cases} \Rightarrow \begin{cases} S = 2 \\ a^3 + b^3 = S^3 - 3SP = 8 - 6P \end{cases}$

$$\frac{1}{a^3} + \frac{1}{b^3} + 12ab \geq 14 \Leftrightarrow \frac{a^3 + b^3}{a^3 b^3} + 12ab \geq 14$$

$$\Leftrightarrow \frac{S^3 - 3SP}{P^3} + 12P \geq 14 \Leftrightarrow \frac{8 - 6P}{P^3} + 12P \geq 14$$

$$8 - 6P + 12P^4 \geq 14P^3, \qquad 12P^4 - 14P^3 - 6P + 8 \geq 0$$

$$6P^4 - 7P^3 - 3P + 4 \geq 0, \qquad 6P^4 - 6P^3 - P^3 + P^2 - P^2 + P - 4P + 4 \geq 0$$

$$6P^3(P - 1) - P^2(P - 1) - P(P - 1) - 4(P - 1) \geq 0$$

$$(P - 1)(6P^3 - P^2 - P - 4) \geq 0$$

$$(P - 1)[(6P^3 - 6P^2) + (5P^2 - 5P) + (4P - 4)] \geq 0$$

$$(P - 1)[6P^2(P - 1) + 5P(P - 1) + 4(P - 1)] \geq 0, \ (P - 1)^2(6P^2 + 5P + 4) \geq 0$$

(true)

$a, b > 0 \Rightarrow ab > 0 \Rightarrow P > 0 \Rightarrow 6P^2 + 5P + 4 > 0$ Equality holds for $a = b = 1$.