ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ and $ab + bc + ca > 0$ then prove that :

$$\sum_{cyc} \frac{1}{a^2 + bc} \geq \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly one among $a, b, c = 0$ and WLOG we may assume $a = 0$ $(b, c > 0)$ and then : $\sum_{cyc} \frac{1}{a^2 + bc} \overset{?}{\geq} \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$

$$\Leftrightarrow \frac{1}{bc} + \frac{1}{b^2} + \frac{1}{c^2} \overset{?}{\geq} \frac{3(b + c)}{bc(b + c)} \Leftrightarrow \left(\frac{1}{b} - \frac{1}{c}\right)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore \sum_{cyc} \frac{1}{a^2 + bc} \geq \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

$\boxed{\text{Case 2}}$ $a, b, c > 0$ and then, assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z,$ $y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $abc = r^2 s, \sum_{cyc} a = s, abc = r^2 s,$

$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^3 = s^3 - 12Rrs,$$

$$\sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2), \sum_{cyc} a^3b^3 = (4Rr + r^2)^3 - 12Rr^3s^2 \text{ and then :}$$

$$\sum_{cyc} \frac{1}{a^2 + bc} \overset{?}{\geq} \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

$$\Leftrightarrow \frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) + \sum_{cyc} a^2b^2}{abc \sum_{cyc} a^3 + \sum_{cyc} a^3b^3 + 2a^2b^2c^2} \overset{?}{\geq} \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

$$\Leftrightarrow \frac{(4Rr + r^2)(s^2 - 8Rr - 2r^2) + r^2((4R + r)^2 - 2s^2)}{r^2 s(s^3 - 12Rrs) + (4Rr + r^2)^3 - 12Rr^3s^2 + 2r^4s^2} \overset{?}{\geq} \frac{3s}{4Rrs}$$

$$\Leftrightarrow -3s^4 + (16R^2 + 68Rr - 6r^2)s^2 - r(256R^3 + 176R^2r + 40Rr^2 + 3r^3) \boxed{\overset{?}{\geq}_{(1)}} 0$$

Now, Rouche $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$, where

$$m = 2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r).\sqrt{R^2 - 2Rr}$$

$$\therefore \left(s^2 - (m + n)\right)\left(s^2 - (m - n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \leq 0$$

$$\Rightarrow P = -3(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \geq 0$$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\overset{?}{\geq}$ P

$$\Leftrightarrow (R+2r)s^2 \boxed{\overset{?}{\underset{②}{\geq}}} r(4R+r)^2 \text{ and finally}, (R+2r)s^2 \overset{\text{Gerretsen}}{\geq} (R+2r)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(4R+r)^2 \Leftrightarrow r^2(19R-11r) \overset{?}{\geq} 0 \to \text{true (strict inequality)} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sum_{\text{cyc}} \frac{1}{a^2+bc} > \frac{3(a+b+c)}{(a+b)(b+c)(c+a)} \; \forall\, a,b,c > 0 \text{ and so,}$$

$$\sum_{\text{cyc}} \frac{1}{a^2+bc} \geq \frac{3(a+b+c)}{(a+b)(b+c)(c+a)} \; \forall\, a,b,c \geq 0,$$

$$'' = '' \text{ iff } \begin{pmatrix} a=0 \\ b=c>0 \end{pmatrix} \text{ or } \begin{pmatrix} b=0 \\ c=a>0 \end{pmatrix} \text{ or } \begin{pmatrix} c=0 \\ a=b>0 \end{pmatrix} \text{ (QED)}$$