ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then prove that :

$$\sum_{cyc} \frac{a}{2a+b+c} \ge \frac{ab+bc+ca}{2(a+b+c)} \cdot \sum_{cyc} \frac{1}{a+b}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0,$
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s \Rightarrow a = s-x, b = s-y, c = s-z,$

$\sum_{cyc} ab = 4Rr + r^2$, and then : $\sum_{cyc} \frac{a}{2a+b+c} - \frac{ab+bc+ca}{2(a+b+c)} \cdot \sum_{cyc} \frac{1}{a+b}$

$$= \sum_{cyc} \frac{s-x}{y+z} - \frac{4Rr+r^2}{2s} \cdot \frac{s^2+4Rr+r^2}{4Rrs}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\sum_{cyc} \left(x^2(s-x)\right) + \left(\sum_{cyc} xy\right)\left(\sum_{cyc}(s-x)\right)\right) - \frac{4Rr+r^2}{2s} \cdot \frac{s^2+4Rr+r^2}{4Rrs}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \begin{pmatrix} 2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2) \\ +s(s^2+4Rr+r^2) \end{pmatrix}$$

$$- \frac{4Rr+r^2}{2s} \cdot \frac{s^2+4Rr+r^2}{4Rrs} = \frac{s^2+8Rr+5r^2}{2(s^2+2Rr+r^2)} - \frac{(4R+r)(s^2+4Rr+r^2)}{8Rs^2} \overset{?}{\ge} 0$$

$$\Leftrightarrow (8R^2+6Rr-2r^2)s^2 - r(32R^3+32R^2r+10Rr^2+r^3) \boxed{\underset{(*)}{\overset{?}{\ge}}} s^4$$

$$\text{Now, } s^4 \overset{\text{Gerretsen}}{\le} (4R^2+4Rr+3r^2)s^2 \overset{?}{\le} \text{LHS of } (*)$$

$$\Leftrightarrow (4R^2+2Rr-5r^2)s^2 \boxed{\underset{(**)}{\overset{?}{\ge}}} r(32R^3+32R^2r+10Rr^2+r^3)$$

Again, $(4R^2+2Rr-5r^2)s^2 \overset{\text{Gerretsen}}{\ge} (4R^2+2Rr-5r^2)(16Rr-5r^2) \overset{?}{\ge}$
RHS of $(**) \Leftrightarrow 8R^3-5R^2r-25Rr^2+r^3 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(8R^2+11Rr-3r^2) \overset{?}{\ge} 0$
$\rightarrow$ true $\because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \Rightarrow (*)$ is true and so,

$$\sum_{cyc} \frac{a}{2a+b+c} \ge \frac{ab+bc+ca}{2(a+b+c)} \cdot \sum_{cyc} \frac{1}{a+b} \; \forall\, a, b, c > 0, '' = '' \text{ iff } a = b = c \text{ (QED)}$$