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If $a, b, c > 0$ then prove that :

$$\sum_{cyc} \frac{(a+b)^3}{c} \geq \sum_{cyc} \frac{(b+c)^3}{c}$$

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$$\sum_{cyc} \frac{(a+b)^3}{c} \overset{?}{\geq} \sum_{cyc} \frac{(b+c)^3}{c}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^3}{c} + \sum_{cyc} \frac{b^3}{c} + 3\sum_{cyc} \frac{a^3b^2 + a^2b^3}{abc} \overset{?}{\geq} \sum_{cyc} \frac{b^3}{c} + \sum_{cyc} a^2 + 3\sum_{cyc} a^2 + 3\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} \frac{a^3}{c} + \frac{3}{abc}\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right) - abc\left(\sum_{cyc} ab\right)\right) \overset{?}{\geq} 4\sum_{cyc} a^2 + 3\sum_{cyc} ab$$

$$and \because \sum_{cyc} \frac{a^3}{c} = \sum_{cyc} \frac{a^4}{ca} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} ab} \geq \sum_{cyc} a^2 \therefore \text{it suffices to prove :}$$

$$\frac{3}{abc}\cdot\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right) - 3\sum_{cyc} ab \overset{?}{\geq} 3\sum_{cyc} a^2 + 3\sum_{cyc} ab$$

$$\Leftrightarrow \frac{3}{abc}\cdot\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right) \overset{?}{\geq} 3\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab\right) = 3\left(\sum_{cyc} a\right)^2$$

$$\Leftrightarrow \sum_{cyc} a^2b^2 \overset{?}{\geq} abc\left(\sum_{cyc} a\right) \rightarrow \text{true} \because A^2 + B^2 + C^2 \geq AB + BC + CA \text{ with}$$

$$A \equiv ab, B \equiv bc, C \equiv ca \therefore \sum_{cyc} \frac{(a+b)^3}{c} \geq \sum_{cyc} \frac{(b+c)^3}{c} \ \forall a, b, c > 0,$$

$$'' = '' \text{ iff } a = b = c \text{ (QED)}$$