ROMANIAN MATHEMATICAL MAGAZINE

**If $a,b,c>0, a+b+c=3$ then:**

$$\sum\frac{a}{b}+\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\geq 3+\sqrt[3]{9}$$

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**Solution by Tapas Das-India**

$$\sum\frac{a}{b}=\sum\frac{a^2}{ab}\geq\frac{(a+b+c)^2}{ab+bc+ca}\overset{q=ab+bc+ac}{=}\frac{9}{q}\ (1)$$

$$q=ab+bc+ca\leq\frac{(a+b+c)^2}{3}\overset{a+b+c=3}{=}3\ \&$$

$$q^2=(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc\overset{r=abc}{=}9r\ or,\frac{q^2}{3}\geq r(2)$$

$\mathit{Lemma}: \forall\, x>0\,,\dfrac{1}{\sqrt[3]{x}}\geq\dfrac{1}{\sqrt[3]{3}}-\dfrac{x-3}{3^{\frac{7}{3}}}$

$\mathit{Proof}: Let\ t=\sqrt[3]{\dfrac{x}{3}}>0\ then\ x=3t^3$

$$so,\frac{1}{\sqrt[3]{x}}\geq\frac{1}{\sqrt[3]{3}}-\frac{x-3}{3^{\frac{7}{3}}}\ or,\frac{1}{\sqrt[3]{3t}}\geq\frac{1}{\sqrt[3]{3}}-\frac{3t^3-3}{3^{\frac{7}{3}}}\quad or,\frac{1}{t}\geq 1-\frac{t^3-1}{3}$$

$$or, t^4-4t+3\geq 0\ or,(t-1)^2(t^2+2t+3)\geq 0\ true$$

$$2(a^3+b^3+c^3)+3abc-9$$
$$=2((a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc))\ +3abc-9$$
$$=9(5-2q+r)\ (putting\ ,q=ab+bc+ca,r=abc,a+b+c=3)$$

$$\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\overset{lemma}{\geq}\frac{3}{\sqrt[3]{3}}-\frac{2(a^3+b^3+c^3)+3abc-9}{3^{\frac{7}{3}}}=\frac{3}{\sqrt[3]{3}}-\frac{9(5-2q+r)}{3^{\frac{7}{3}}}$$

$$L.H.S-(3+\sqrt[3]{9})=$$

$$=\frac{9}{q}+\left(\frac{3}{\sqrt[3]{3}}-\frac{9(5-2q+r)}{3^{\frac{7}{3}}}\right)-(3+\sqrt[3]{9})\overset{\frac{q^2}{3}\geq r}{\geq}\frac{3(3-q)}{q}-\frac{(45-18+q^2)3^{\frac{2}{3}}}{27}$$

$$=\frac{(3-q)\left(81+(q-15)3^{\frac{2}{3}}\right)}{27q}\geq 0\ \left(as\ q\leq 3\ so\ 3-q>0\ and\ 81-(q-15)3^{\frac{2}{3}}>0\right)$$

$$soL.H.S-(3+\sqrt[3]{9})\geq 0\ or,\sum\frac{a}{b}+\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\geq 3+\sqrt[3]{9}$$

**Equality holds for a=b=c=1.**