ROMANIAN MATHEMATICAL MAGAZINE

If $x,y>0$ and $xy=4$ then prove that:

$$\frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}}\ge\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^{\frac{3}{2}}+y^{\frac{3}{2}}=\frac{x^{\frac{3}{2}}}{(1)^{\frac{1}{2}}}+\frac{y^{\frac{3}{2}}}{(1)^{\frac{1}{2}}}\overset{Radon}{\ge}\frac{(x+y)^{\frac{3}{2}}}{(1+1)^{\frac{1}{2}}}\overset{x+y=t}{=}\frac{t^{\frac{3}{2}}}{\sqrt{2}}\ (1)\&\ t=x+y\ge 2\sqrt{xy}\overset{xy=4}{=}\ 4\ (2)$$

$$\sqrt{x^3+1}+\sqrt{y^3+1}=\sqrt{\left(x^{\frac{3}{2}}\right)^2+(1)^2}+\sqrt{\left(y^{\frac{3}{2}}\right)^2+(1)^2}\overset{Minkowski}{\ge}$$

$$\ge\sqrt{\left(x^{\frac{3}{2}}+y^{\frac{3}{2}}\right)^2+(1+1)^2}\overset{(1)}{\ge}\sqrt{\frac{t^3}{2}+4}$$

$$\sqrt{x^3+1}.\sqrt{y^3+1}=\sqrt{(x^3+1)(y^3+1)}=\sqrt{1+x^3+y^3+(xy)^3}=$$

$$=\sqrt{1+(x+y)^3-3xy(x+y)+(xy)^3}\overset{\substack{xy=4\\x+y=t}}{=}\sqrt{65-12t+t^3}$$

$$\frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}}=\frac{\sqrt{x^3+1}+\sqrt{y^3+1}}{\sqrt{x^3+1}.\sqrt{y^3+1}}\overset{(1)}{\ge}\frac{\sqrt{\frac{t^3}{2}+4}}{\sqrt{65-12t+t^3}}$$

$$We\ need\ to\ show:\frac{\sqrt{\frac{t^3}{2}+4}}{\sqrt{65-12t+t^3}}\ge\frac{2}{3}\ or,3\sqrt{\frac{t^3}{2}+4}\ \ge 2\sqrt{65-12t+t^3}$$

$$or,9\left(\frac{t^3}{2}+4\right)\ge 4(65-12t+t^3)\ or,t^3+96t-448\ge 0$$

$$or,(t-4)(t^2+4t+112)\ge 0\ true\ as\ t\ge 4\ by\ (2)$$

Equality holds for x=y=2