ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x + y + z + 2 = xyz$ then prove that :

$$\frac{1}{\sqrt{x^3+1}} + \frac{1}{\sqrt{y^3+1}} + \frac{1}{\sqrt{z^3+1}} \geq 1$$

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Let $x = A + 1, y = B + 1, z = C + 1$ and then : $A + B + C + 5 = (A + 1)(B + 1)(C + 1) \Rightarrow ABC + AB + BC + CA = 4$

$$\Rightarrow \sum_{cBc}\big((2+B)(2+C)\big) - (2+A)(2+B)(2+C) = 4 - (AB + BC + CA + ABC)$$

$$= 0 \therefore \sum_{cBc} \frac{1}{2+A} = 1 \rightarrow (m)\ (\because 2 + A = x + 1 > 0 \text{ and analogs}) \text{ and setting :}$$

$$\frac{1}{2+A} = \frac{1}{2} - \alpha \Rightarrow A = \frac{2\alpha}{\frac{1}{2} - \alpha} \rightarrow (1); \text{Similarly, setting : } \frac{1}{2+B} = \frac{1}{2} - \beta \text{ and}$$

$$\frac{1}{2+C} = \frac{1}{2} - \gamma \Rightarrow 1 \overset{\text{via (m)}}{=} \frac{1}{2} - \alpha + \frac{1}{2} - \beta + \frac{1}{2} - \gamma \Rightarrow \alpha + \beta + \gamma = \frac{1}{2} \rightarrow (i)$$

$$\beta + \gamma = \frac{1}{2} - \alpha = \frac{1}{2+A} > 0\ ; \text{similarly}, \gamma + \alpha > 0\ \&\ \alpha + \beta > 0 \therefore (1), (i) \Rightarrow$$

$$A = \frac{2\alpha}{\beta + \gamma} \text{ and analogously}, B = \frac{2\beta}{\gamma + \alpha} \text{ and } C = \frac{2\gamma}{\alpha + \beta} \text{ and via such}$$

$$\text{substitutions}, x = A + 1 = \frac{2\alpha}{\beta + \gamma} + 1 = \frac{(\gamma + \alpha) + (\alpha + \beta)}{\beta + \gamma} = \frac{b + c}{a}$$

$$(a = \beta + \gamma > 0; b = \gamma + \alpha > 0; c = \alpha + \beta > 0); \text{similarly}, y = \frac{c + a}{b} \text{ and } z = \frac{a + b}{c}$$

Assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0\ \&\ Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{then : } \sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2,$$

$$\sum_{cyc} a^3 = s^3 - 12Rrs, \sum_{cyc} a^2 b^2 = r^2\big((4R + r)^2 - 2s^2\big)$$

$$\therefore \sum_{cyc} \frac{1}{\sqrt{x^3+1}} = \sum_{cyc}\left(\frac{1}{\sqrt{\frac{b+c}{a} + 1}} \cdot \frac{1}{\sqrt{\left(\frac{b+c}{a}\right)^2 - \frac{b+c}{a} + 1}}\right)$$

$$= \frac{1}{\sqrt{\sum_{cyc} a}} \cdot \sum_{cyc} \frac{a.\sqrt{a}.a^2.\sqrt{a}}{a^2.\sqrt{a}.\sqrt{(b+c)^2 - a(b+c) + a^2}}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{1}{\sqrt{\sum_{cyc} a}} \cdot \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc}\left(a^2.\sqrt{a}.\sqrt{(b+c)^2 - a(b+c) + a^2}\right)}$$

$$\overset{CBS}{\geq} \frac{1}{\sqrt{\sum_{cyc} a}} \cdot \frac{\left(\sum_{cyc} a^2\right)^2}{\sqrt{\sum_{cyc} a^3} \cdot \sqrt{2\sum_{cyc} a^2b^2 + 2abc\sum_{cyc} a - \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) + abc\sum_{cyc} a + \left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2}} \overset{?}{\geq} 1$$

$$\Leftrightarrow (s^2 - 8Rr - 2r^2)^4 \overset{?}{\geq} s(s^3 - 12Rrs)\left(3r^2s^2 + (s^2 - 8Rr - 2r^2)(s^2 - 12Rr - 3r^2)\right)$$

$$\Leftrightarrow -3s^6 + (24R^2 + 60Rr + 9r^2)s^4 - r(448R^3 + 480R^2r + 156Rr^2 + 16r^3)s^2 + 8r^2(4R + r)^4 \boxed{\overset{?}{\geq}_{①}} 0 \text{ and } \because P = -3s^2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3)$$

$$\overset{Double-Rouche}{\geq} 0 \therefore \text{to prove ①, suffices to prove : } LHS_{①} \overset{?}{\geq} P \Leftrightarrow (12R^2 + 15r^2)s^4 - r(256R^3 + 336R^2r + 120Rr^2 + 13r^3)s^2 + 8r^2(4R + r)^4 \boxed{\overset{?}{\geq}_{②}} 0 \text{ and } \because$$

$$(12R^2 + 15r^2)(s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{to prove ①, it suffices to prove :}$$

$$\text{LHS of ② } \overset{?}{\geq} (12R^2 + 15r^2)(s^2 - 16Rr + 5r^2)^2 \Leftrightarrow$$

$$(X = 128R^3 - 456R^2r + 360Rr^2 - 163r^3)s^2 \boxed{\overset{?}{\geq}_{③}} r(1024R^4 - 3968R^3r + 3372R^2r^2 - 2528Rr^3 + 367r^4)$$

$\boxed{\textit{Case 1}}$ $X \geq 0$ *and* then : LHS of ③ $\overset{Gerretsen}{\geq} X(16Rr - 5r^2) \overset{?}{\geq}$ RHS of ③

$$\Leftrightarrow 256t^4 - 992t^3 + 1167t^2 - 470t + 112 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(256t^3 - 480t^2 + 207t - 56) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow \text{③ is true}$$

$\boxed{\textit{Case 2}}$ $X < 0$ *and* then : LHS of ③ $\overset{Gerretsen}{\geq} X(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of ③

$$\Leftrightarrow 128t^5 - 584t^4 + 992t^3 - 988t^2 + 739t - 214 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(128t^3 - 72t^2 + 192t + 68) + 243\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow$ ③ is true $\therefore$ combining both *cases*, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true $\forall \Delta XYZ_{s,R,r}$

$$\therefore \frac{1}{\sqrt{x^3+1}} + \frac{1}{\sqrt{y^3+1}} + \frac{1}{\sqrt{z^3+1}} \geq 1 \;\forall\, x, y, z > 0 \,\middle|\, x + y + z + 2 = xyz,$$

$'' = ''$ iff $x = y = z = 2$ (QED)