ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ then prove that :

$$\frac{2a^3+3b^3}{2a^4+3b^4}+\frac{2b^3+3a^3}{2b^4+3a^4}\le\frac{4}{a+b}$$

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$$\frac{2x^3+3y^3}{2x^4+3y^4}=\frac{(2x^3+3y^3)(2x^2+3y^2)}{(2x^4+3y^4)(2x^2+3y^2)}\overset{\text{Reverse CBS}}{\le}\frac{(2x^3+3y^3)(2x^2+3y^2)}{(2x^3+3y^3)^2}$$

$$=\frac{(2x^2+3y^2)(2x+3y)}{(2x^3+3y^3)(2x+3y)}\overset{\text{Reverse CBS}}{\le}\frac{(2x^2+3y^2)(2x+3y)}{(2x^2+3y^2)^2}$$

$$\therefore\frac{2x^3+3y^3}{2x^4+3y^4}\le\frac{2x+3y}{2x^2+3y^2}\to (1)$$

Substituting $x\equiv a$ and $y\equiv b$ in (1), we get : $\frac{2a^3+3b^3}{2a^4+3b^4}\le\frac{2a+3b}{2a^2+3b^2}\to (2)$

Substituting $x\equiv b$ and $y\equiv a$ in (1), we get : $\frac{2b^3+3a^3}{2b^4+3a^4}\le\frac{2b+3a}{2b^2+3a^2}\to (3)$

$$\therefore (2)\text{ and }(3)\Rightarrow\text{it suffices to prove : }\frac{2a+3b}{2a^2+3b^2}+\frac{2b+3a}{2b^2+3a^2}\overset{?}{\le}\frac{4}{a+b}$$

$$\Leftrightarrow 4(2a^2+3b^2)(2b^2+3a^2)\overset{?}{\ge}(a+b)\begin{pmatrix}(2a+3b)(2b^2+3a^2)+\\(2b+3a)(2a^2+3b^2)\end{pmatrix}$$

$$\Leftrightarrow 12t^4-25t^3+26t^2-25t+12\overset{?}{\ge}0\ \left(t=\frac{a}{b}\right)$$

$$\Leftrightarrow (t-1)^2\left((t-1)^2+11t^2+t+11\right)\overset{?}{\ge}0\to\text{true}\because t>0$$

$$\therefore\frac{2a^3+3b^3}{2a^4+3b^4}+\frac{2b^3+3a^3}{2b^4+3a^4}\le\frac{4}{a+b}\ \forall a,b>0,''=''\text{ iff }a=b\ (QED)$$